

Kink-soliton explosions in generalized Klein–Gordon equations

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Abstract

We investigate the dynamics of kink-solitons in generalized Klein–Gordon equations in the presence of nonlinear damping and spatiotemporal perturbations. We present different mechanisms for kink-soliton explosion and show that, in some cases while some analytically obtained conditions are satisfied, the kink-soliton breaks up becoming a permanent, extremely complex, spatiotemporal dynamics. We perform computer simulations in order to visualize and corroborate our analytical findings. The mechanisms presented for kink-soliton breakup can explain some of the phenomena that recently have been reported to occur in excitable media. Finally, we will present a way to control soliton breakup, using appropriate external perturbations.

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1. Introduction

Solitons are self-localized solutions of nonlinear evolution equations [1–3]. One of the most attractive characteristics of solitons is that they can propagate without visible changes. However, under some conditions, solitons may become unstable [4–16]. Such instabilities had been called soliton breakup or soliton explosion.

Milchev and coworkers have studied the Frenkel–Kontorova model with anharmonic interatomic interaction [4–8] and found that a breakup of the kink takes place when the effective potential amplitude exceeds a certain critical value. An extensive discussion of soliton dynamics in the Frenkel–Kontorova model can be found in the recent book [17]. On the other hand, in Ref. [13,14] it was predicted that instability in the soliton internal mode leads to soliton breakup.

Kink-solitons [1–3,18], like vortices and spirals [19–21], are particular cases of a more general phenomenon called topological defects. Although these objects may possess different origin and nature in different physical systems, they all possess very similar dynamical properties.

The study of spiral breakup has been a very active area of research in recent years [20]. Spiral breakup has been observed in experiments and simulations [22–30].

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One possible mechanism which is currently believed to be responsible for the transition from tachycardia to ventricular fibrillation is the spontaneous breakup of a single spiral wave of electrical activity into multiple spirals leading to a turbulent wave behavior [27].

The investigation of the transition from regular defect motion to highly nonstationary spatiotemporal dynamics remains a challenge in modern science [31]. This phenomenon is related to defect-mediated turbulence [31,32].

We will present different mechanisms for soliton breakup and explosions. We find that in some cases, while some conditions hold, the soliton explosion is permanent. That is, if the conditions for soliton explosions have not changed, the soliton does not return to the original steady state. In some cases, this dynamics is very similar to defect turbulence.

In the present paper, we investigate generalized Klein–Gordon equations as the following

$$\phi_{tt} + R(\phi_t) - \phi_{xx} - G(\phi) = F(x, t), \quad (1)$$

where $G(\phi) = -\partial U(\phi)/\partial \phi$, $U(\phi)$ is a potential function with at least two minima ϕ_1 , ϕ_3 and a maximum ϕ_2 , such that $U(\phi_1) = U(\phi_3)$, $R(\phi_t)$ are dissipative terms, and $F(x, t)$ represents external perturbations. We are interested in kinks, that is, topological solitons between the points ϕ_1 and ϕ_3 . The famous sine-Gordon and ϕ^4 -systems are particular cases of Eq. (1).

The topological solitons studied in the present paper possess important applications in condensed matter physics. For instance, in solid state physics, they describe domain walls in ferromagnets and ferroelectric materials, dislocations in crystals, charge-density waves, interphase boundaries in metal alloys, fluxons in long Josephson junctions and Josephson transmission lines, etc. [1].

Such instabilities as soliton breakups can affect all mentioned applications. Therefore it is very important to understand all the possible mechanisms of soliton breakup in order to avoid and control [33] them.

In the following, we present the stability of kink-soliton solutions in Section 2, different mechanisms for soliton breakup in Section 3, and show that in some cases, while some conditions hold, the soliton becomes a permanent, extremely complex, nonstationary spatiotemporal dynamics. In Section 4, we present a way to control such breakups from happening. Conclusions are given at the end of the paper.

2. Stability of kink-solitons

It is well known that the existence of heteroclinic trajectories joining the fixed points of the corresponding dynamical system [34], is a condition for the existence of kink-soliton solutions. Therefore we perform a complete investigation of equations of type (1) using the so-called qualitative theory of dynamical systems [35,36] (including topological concepts), and with the additional information about the behavior of solutions in the neighborhood of fixed points and separatrices, it is possible to construct functions with the general properties of the exact solutions of the given equation. Then, solving an inverse problem, we are able to find exact solution for systems of type (1). Later, it is possible to generalize the results to some classes of equations that are topologically equivalent to those with the exact solutions.

In some cases, it is sufficient to have an exact solution corresponding to some relevant physical situation and for which we can solve exactly the stability problem.

For instance, let us study the following equation

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} - G(\phi) = F(x). \quad (2)$$

We are interested in the stability of kinks at the equilibrium positions created by the inhomogeneous force $F(x)$. Constructing a general function $\phi_k(x)$ with the topological and general properties of a kink, we solve an inverse problem such that $F(x)$ possesses the properties of the physical system we are studying. Now we can investigate the stability of the solution $\phi(x, t) = \phi_k(x) + f(x)e^{\lambda t}$. For this we solve the spectral problem

$$\hat{L}f(x) = \Gamma f(x), \quad (3)$$

where

$$\hat{L} = -\partial_{xx} - \left(\frac{\partial G(\phi)}{\partial \phi} \right)_{\phi=\phi_k(x)},$$

$$\Gamma = -(\lambda^2 + \gamma\lambda).$$

The results obtained with this function can be generalized to other systems topologically equivalent to the exactly solvable system [9–12,18,37–39].

Let us see the following example:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = F_1(x), \tag{4}$$

where $F_1(x) = 2(B^2 - 1)\sinh(Bx)/\cosh^2(Bx)$ is a localized force.

The exact stationary solution of Eq. (4) is $\phi_k(x) = 4\arctan[\exp(Bx)]$. This solution represents a kink-soliton at the position $x = 0$. The stability problem (3) can be solved exactly. The eigenvalues of the discrete spectrum are given by the formula

$$\Gamma_n = B^2(\Lambda + 2\Lambda n - n^2) - 1, \tag{5}$$

where $\Lambda(\Lambda + 1) = 2/B^2$.

The integer part of Λ , i.e. $[\Lambda]$, yields the number of eigenvalues in the discrete spectrum, which correspond to the soliton modes (this includes the translational mode Γ_0 , and the internal and shape modes Γ_n ($0 < n < [\Lambda]$). Everything concerning the stability of the soliton in this situation can be obtained from the equation for Γ_n .

3. Kink-soliton breakup mechanisms

We wish to discuss the following situations where the kink-soliton can be destroyed: constant external perturbation, inhomogeneous space-dependent perturbation, time-dependent perturbations, and nonlinear damping. We will use the sine-Gordon and/or the ϕ^4 equations to see what would be arising from each case.

3.1. Constant external perturbation

The simplest case is when a constant external perturbation, whose magnitude exceeds some critical value, is applied:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} - G(\phi) = F_0. \tag{6}$$

For the existence of kink solutions we need that the fixed points of the corresponding dynamical system be joined by heteroclinic trajectories [34]. That is, the effective potential $U_{\text{eff}} = U(\phi) - F_0\phi$ has at least two minima. Define F_c as the value where the above condition cease to happen.

When $|F_0| < F_c$, a kink, which is initially at rest, will accelerate until it reaches a constant velocity that is the result of the balance between the external force and the damping [40], see Fig. 1. On the other hand, if $|F_0| > F_c$ then, the kink is destroyed as shown in Figs. 2 and 3. We will call F_c the critical value for the destruction to occur.

In Fig. 3 the asymptotic spatiotemporal state is just a spatially homogeneous structure that corresponds to the only persistent minimum of $U_{\text{eff}}(\phi)$.

3.2. Inhomogeneous space-dependent perturbations

A slightly different kink-soliton destruction is observed when inhomogeneous space-dependent perturbations are present:

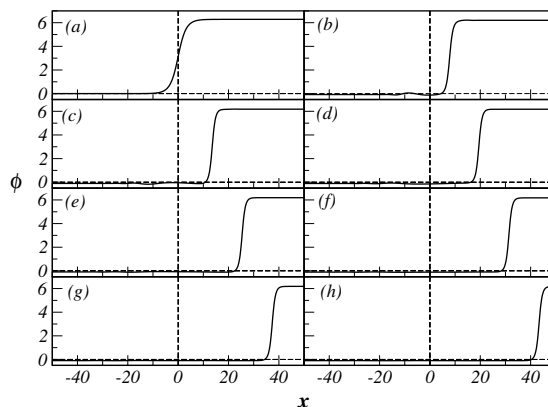


Fig. 1. The sine-Gordon kink-soliton can be driven by a small constant external force (see Eq. (6)). Due to dissipation, in the long-time behavior, the kink will move with a constant velocity. Here $G(\phi) = -\sin(\phi)$, $\gamma = 0.1$, $F_0 = -0.1$.

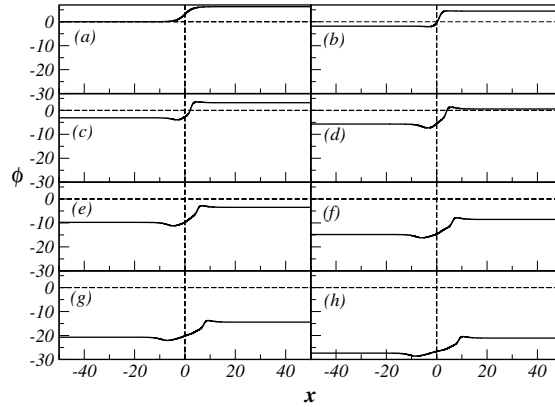


Fig. 2. Destruction of the sine-Gordon kink using a constant external force ($|F_0| > F_c$) in Eq. (6) in the presence of dissipation. Here $\gamma = 0.1$, as in Fig. 1, but $F_0 = -1.001$. The solution $\phi(x,t)$ vanishes since there is no solution of the algebraic equation $\sin(\phi) = F_0$.

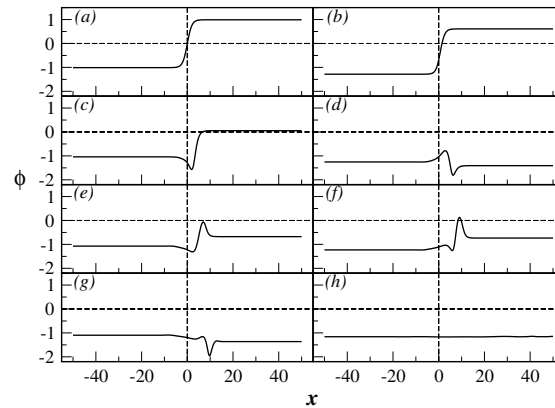


Fig. 3. Destruction of the ϕ^4 kink using a constant external force ($|F_0| > F_c$) in Eq. (6) in the presence of dissipation. Here $G(\phi) = (\phi - \phi^3)/2$, $\gamma = 0.1$, $F_0 = -0.2$. Note that “nothing” is left. The solution $\phi(x,t)$ is a constant function where $\phi = \phi_1$ is the only real solution of the algebraic equation $-\frac{(\phi - \phi^3)}{2} = F_0$.

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} - G(\phi) = F(x). \tag{7}$$

We should mention that the zeroes of $F(x)$ are candidates for equilibrium positions for the kink [9,11,10,12,39]. If $F(x)$ possesses only one zero x_0^* ($F(x_0^*) = 0$), then it is a stable position for the kink if $(\partial F/\partial x)_{x_0^*} > 0$. Otherwise, the position is an unstable equilibrium (while the opposite is true for the anti-kink). The center of mass of a kink can oscillate around a stable zero of $F(x)$, and move away from an unstable one.

However, when $F(x)$ has an unstable zero at $x = x_0^*$ and additionally, the following conditions holds $\lim_{x \rightarrow \pm\infty} F(x) = \mp F_\infty$, $F_\infty > 0$ and $F_\infty > F_c$, then the kink can be destroyed and an anti-kink is “created” in the equilibrium position $x = x_0^*$ (see Fig. 4).

When the kink is in an unstable equilibrium position, it is “stretched” by the pair of forces that are acting on its body in opposite directions. And there is a limit for the magnitude of the pair of forces that the kink can resist. This limit is F_c .

Nevertheless, there is a more subtle mechanism for kink-soliton destruction. When a kink is close to an unstable equilibrium position and $F_\infty < F_c$, many internal shape modes of the kink can be excited ($|A| > 1$).

If $(\partial^2 F/\partial x^2)_{x_0^*}$ is larger than some critical value ($\Gamma_1 < 0$), then the first internal shape mode become unstable and this instability can lead to the kink destruction. What is interesting in this situation is that $F(x)$ needs not to be large for large x as in the previous case. $F(x)$ can be a localized perturbation as that shown in Fig. 5.

Figs. 6 and 7 show the case when the first internal shape mode is unstable, causing the kink to decay into an antikink and two kinks.

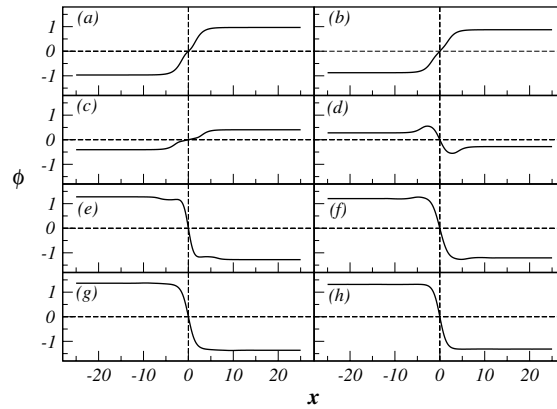


Fig. 4. Instability of the internal structure of the ϕ^4 kink in Eq. (7). An antikink is created in the place of the kink. Here $G(\phi) = (\phi - \phi^3)/2$, $F(x) = A \tanh(Bx)$, $A = -0.5$, $B = 0.707$, $\gamma = 0.4$.

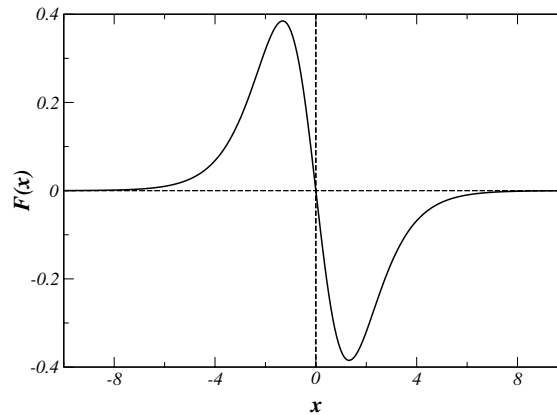


Fig. 5. Localized inhomogeneous external force that can make the kink unstable producing an antikink and two kinks as in Fig. 6.

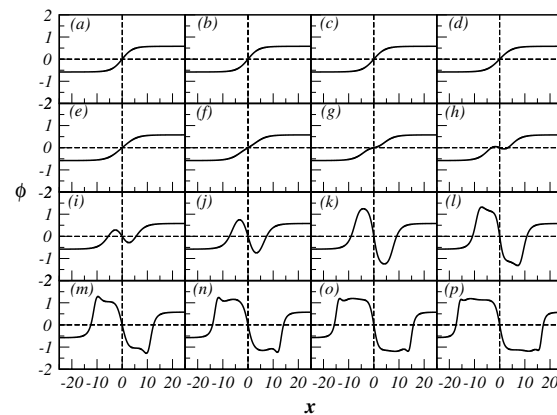


Fig. 6. Instability of the ϕ^4 kink produced by an inhomogeneous external force. The kink decays into an antikink and two kinks that move away from the point $x = 0$. This phenomenon is produced due to the instability of the first shape mode. Here the parameters are the same as in Fig. 4, but $F(x) = B(4B^2 - 1)\tanh(Bx)$, $B = 0.29$. Exactly the same picture can be obtained using a localized force as that represented in Fig. 5.

3.3. Time-dependent perturbations

We should say that a kink, moving in a medium that is homogeneous everywhere except for a zone where the conditions for the instabilities hold, can undergo dramatic transient changes. But when the kink leaves the mentioned zone, it will return to its original steady state shape (see Figs. 6 and 7).

How can we produce a permanent kink-soliton breakup? One answer is to use time-dependent perturbations.

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} - G(\phi) = F_1(x) + F_2(x, t), \tag{8}$$

where $F_1(x)$ is a perturbation that creates a potential well for the kink (i.e. $F_1(x)$ possesses a zero x_0^* such that $\partial F_1/\partial x|_{x_0^*} > 0$) and $F_2(x, t)$ is a space-time perturbation that periodically generates the instability conditions.

Figs. 8–11 show several examples of those highly complex spatio-temporal behaviors due to the instabilities generated by the spatio-temporal perturbation.

3.4. Nonlinear damping

Can we produce permanent kink-soliton breakup without time-dependent external perturbations?

Here we would like to remark that kinks can move with constant velocity (without attenuation) in active and excitable media, and in systems with nonlinear damping even without explicit external forces.

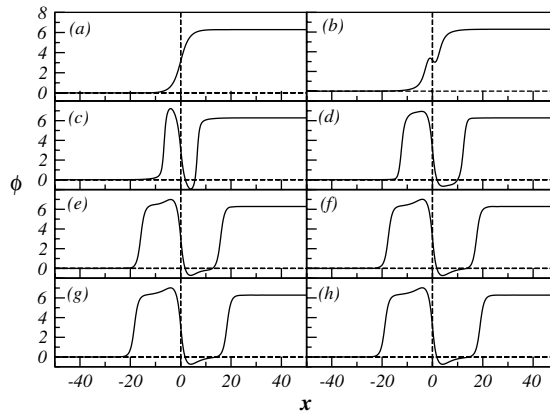


Fig. 7. The same as in Fig. 6, but for the sine-Gordon kink; $G(\phi) = -\sin(\phi)$, $F(x) = 2(B^2 - 1)\tanh(Bx)\text{sech}(Bx)$, $B = 0.301$, $\gamma = 0.2$.

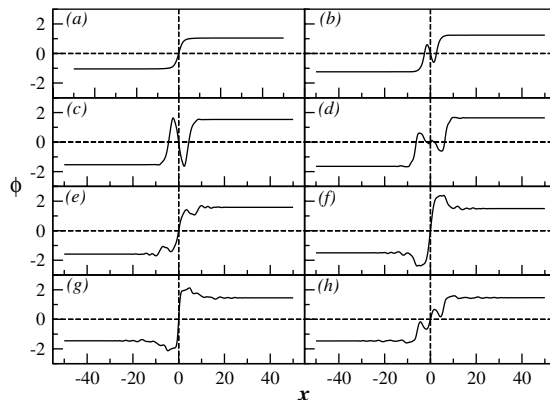


Fig. 8. Kink-soliton breakup sustained by a time dependent perturbation. Note that, at some moment, the kink can be recomposed and then it is destroyed again. Here $G(\phi) = (\phi - \phi^3)/2$, $F_1(x) = (1/2)A(A^2 - 1)\tanh(Bx)$, $F_2(x, t) = (1/2)f_0A(4B^2 - A^2)\cos(\omega t)\sinh(Bx)\cosh^{-3}(Bx)$, $A = 1.5$, $B = 0.2$, $f_0 = 1$, $\omega = 1$, $\gamma = 0.1$.

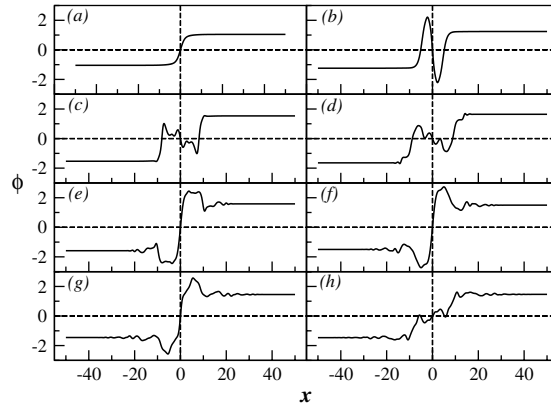


Fig. 9. The same as in Fig. 8 but with different parameter $f_0 = 2$.

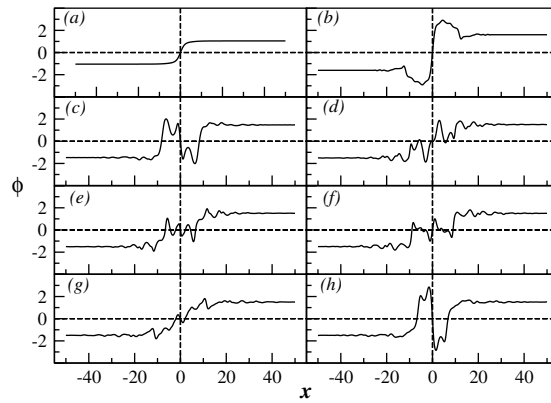


Fig. 10. A more dramatic kink-soliton breakup than in Figs. 8 and 9. The amplitude of the external force is larger: $f_0 = 3$.

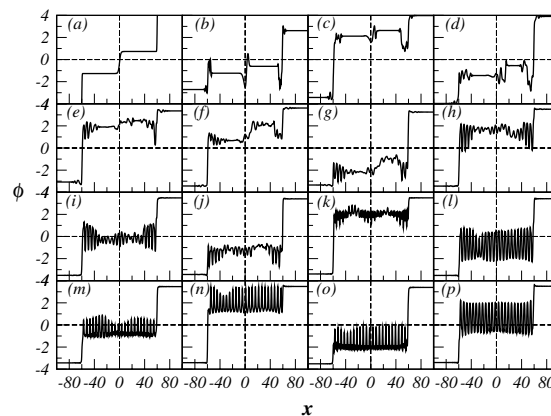


Fig. 11. Highly nonstationary spatiotemporal dynamics after kink-soliton breakup. The difference with Figs. 8–10 is that $F_1(x) = A\{4\arctan[\exp(B(x+d))] + 4\arctan[\exp(B(x-d))] - 2\pi\}$, $F_2(x,t) = f_0 \tanh(Bx) \cos(\omega t)$, $A = 3$, $B = 6$, $D = 1$, $f_0 = 1$, $l = 60$, $\omega = 1$.

Let us discuss here briefly the importance of nonlinear damping. Linear dissipative systems like the damped harmonic oscillator $\phi_{tt} + \gamma\phi_t + \omega_0^2\phi = 0$ cannot sustain oscillations. However, the nonlinear oscillator $\phi_{tt} - b\phi_t + a\phi_t^3 + \omega_0^2\phi = 0$ supports a stable limit cycle [35,36]. The transition from a stable focus to an unstable focus and a stable limit cycle is the result of a Hopf bifurcation. This system is very easy to realize in practice using negative-resistance electronic elements [41].

Kink-soliton bearing systems as the following:

$$\phi_{tt} + R(\phi_t) - \phi_{xx} - G(\phi) = 0, \tag{9}$$

where $dR(\phi_t)/d\phi_t$ is negative for small values of $|\phi_t|$ and positive elsewhere, can support kinks moving with a constant velocity. An example of this kind of systems can be realized in practice using a Josephson junction transmission line where the resistor is a negative-resistance twin-tunnel-diode circuit or a twin-transistor system [41]. In this case, $R(\phi_t) = -b\phi_t + a\phi_t^3$ is a good model.

From Eq. (9) we obtain that kinks that move without changes of shape and velocity correspond to solutions of the equation

$$\phi_{zz} - R(-w\phi_z) + G(\phi) = 0, \tag{10}$$

where $z = (x - vt)/\sqrt{1 - v^2}$ and $w = v/\sqrt{1 - v^2}$, v being a velocity of the kink that satisfies the equation

$$\int_{-\infty}^{\infty} R(-w\phi_z)\phi_z dz = 0. \tag{11}$$

Eq. (11) is satisfied by three values of the velocity v : $v = 0$, $v = v_1 > 0$, and $v = v_2 = -v_1 < 0$. The velocity $v = 0$ is unstable.

However, the solution is a kink-soliton only if function

$$V(\phi) = U(\phi) + \int_{-\infty}^{\infty} R(-w\sqrt{2U(\phi)}) d\phi \tag{12}$$

satisfies the condition $V(\phi) > 0$ in the whole interval $\phi_1 < \phi < \phi_3$. Suppose $R(\phi_t)$ possesses two local extrema: a maximum and a minimum such that the value of $|R(\phi_t)|$ at these extrema is R_m . If this value is comparable with the absolute value of the extrema of $G(\phi)$ (let us call it G_m) in the interval $\phi_1 < \phi < \phi_3$, then the condition $V(\phi) > 0$ may be not satisfied. In fact, if $R_m > G_m$ this condition is certainly not satisfied. When this happens, the kink becomes a highly non-stationary state.

If $R(\phi_t) = -b\phi_t + a\phi_t^3$, then $R_m = 2/3b\sqrt{b/3a}$. So, for a given $G(\phi)$, and a fixed a , parameter b is the key. For small b , the kink can move smoothly with a constant velocity. For larger values of b , a big transformation will occur. This phenomenon can be observed in Figs. 12 and 13.

When $F(x)$ in Eq. (7) has a stable zero, say $x = x_0^*$, the center of mass of a kink can perform damped oscillations around x_0^* . If we wish to sustain these oscillations without explicit time-periodic external forces, then we should resort again to negative damping. Another way to experiment negative damping is when the damping coefficient in Eq. (7) is a function of x :

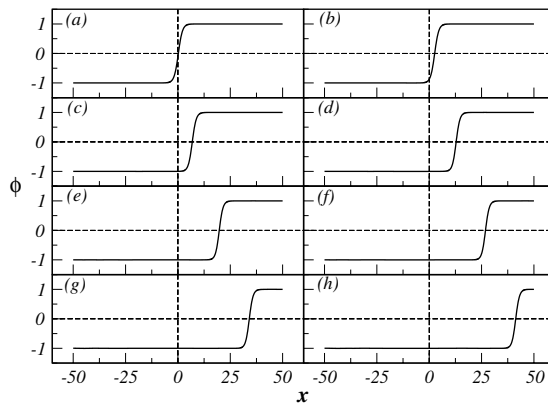


Fig. 12. Despite dissipation, Eq. (9) can sustain a kink with constant velocity due to negative resistance. Here $G(\phi) = (\phi - \phi^3)/2$, $R(\phi_t) = -b\phi_t + a\phi_t^3$, $a = 1$, $b = 0.05$.

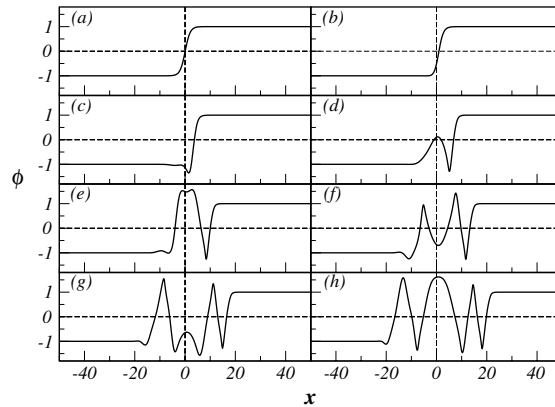


Fig. 13. Kink-soliton breakup due to nonlinear damping. If $R(\phi_t) = -b\phi_t + a\phi_t^3$, and parameter b is larger than some critical value, the kink will explode. The parameters take the same values as in Fig. 1 but $b = 0.7$.

$$\phi_{tt} + \Gamma(x)\phi_t - \phi_{xx} - G(\phi) = F(x). \tag{13}$$

Here $\Gamma(x)$ is negative in a neighborhood of x_0^* and positive elsewhere. This can be done in a chain of nonlinear oscillators using negative-resistance circuits [41] only in some small interval of the chain.

An example of $\Gamma(x)$ with the required features is $\Gamma(x) = \gamma[1 - L/\cosh^2(Dx)]$, where $(1 - L) < 0$.

Fig. 14 shows a limit cycle which is the result of the dynamics of the kink center of mass in Eq. (13). However if we are not careful, the kink can explode also in this system. In fact, if $\Gamma(0) > G_m$, then we can observe a very turbulent behavior as that shown in Fig. 15.

4. Controlling soliton breakup

Can we control all these types of turbulent dynamics? Well, if we can change the parameters of the system, then we can use parameter values that do not lead to soliton breakup. However, sometimes we can not change the parameters. We just are allowed to apply some external perturbation.

Let us pose the following problem:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} - G(\phi) = F_p(x, t) + F_c(x). \tag{14}$$

Eq. (14) represents a system with turbulent behavior as that shown in Figs. 10, 13 and 15, when the control perturbation $F_c(x) = 0$.

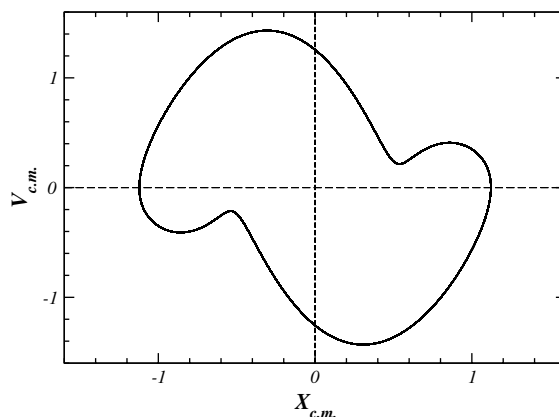


Fig. 14. Limit cycle produced with the dynamics of the kink center of mass in Eq. (13). Here $\Gamma(x) = 1 - l\cosh^{-1}(Bx)$, $G(\phi) = (\phi - \phi^3)/2$, $F(x) = A\tanh(Bx)$, $l = 2$, $D = 0.65$, $A = 0.45$, $B = 0.65$.

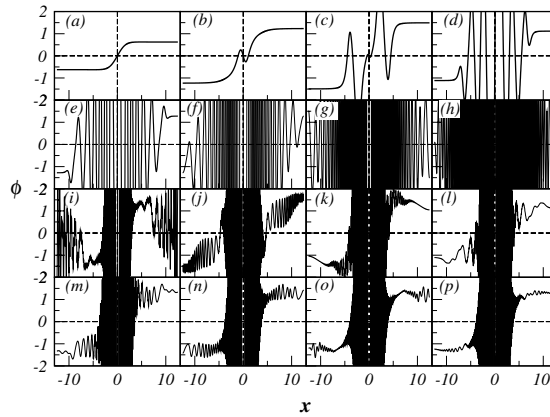


Fig. 15. Highly nonstationary spatiotemporal dynamics produced by Eq. (13) if $\Gamma(0) > G_m$. In this case the parameters are the same as in Fig. 14 but $l = 6$.

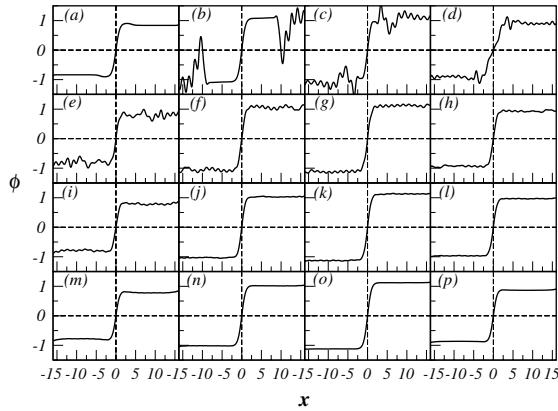


Fig. 16. Controlling soliton breakup. Here the destructive perturbation is $F_p(x, t) = F_1(x) + F_2(x, t)$ as in Fig. 11. The controlling perturbation is $F_c(x) = g \tanh(Dx) \cosh^{-2}(Dx)$, $g = 1.68$. Note that we are controlling the soliton breakup produced by time-dependent extended perturbations using a static localized perturbation.

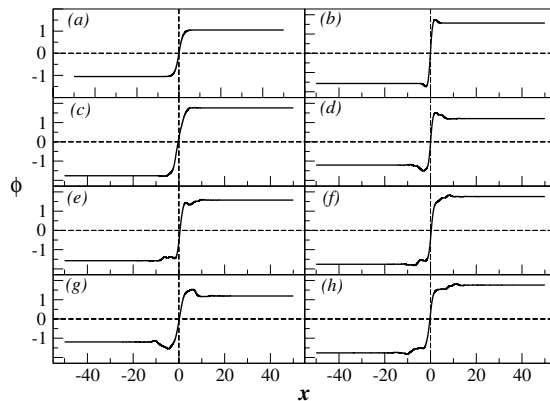


Fig. 17. Controlling the soliton breakup produced by nonlinear damping. Here $G(\phi) = (\phi - \phi^3)/2$, $R(\phi_t) = -b\phi_t + a\phi_t^3$, $F_c(x) = g \tanh(Ex)$, $a = 1$, $b = 0.2$, $E = 1$, $g = 1$.

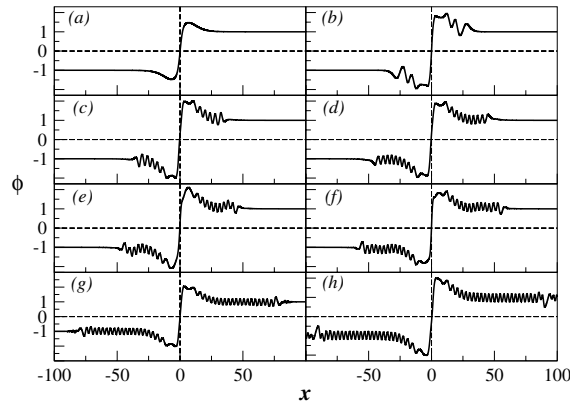


Fig. 18. Controlling soliton breakup produced by nonlinear damping using a localized perturbation. Here all the functions are the same as in Fig. 17, but $F_c(x) = g \tanh(Ex) \cosh^{-2}(Ex)$, $E = 0.1$, $g = 7.0$. Note that the resulting dynamics is not completely stationary but the soliton does not break.

The problem is to find a controlling perturbation $F_c(x)$. Suppose $F_p(x, t)$ is a function that periodically generates the instability conditions discussed in the first part of the paper. An example can be the following $F_p(x, t) = a \cos(\omega t) - \tanh(Bx)$. The strategy could be to find a perturbation $F_c(x)$ such that the superposition $F_p(x, t) + F_c(x)$ does not satisfy the instability condition anymore for any t [42,43].

It is remarkable that this can be achieved using a localized perturbation. For instance $F_p(x, t) = -0.385 \cos(\omega t) - \tanh(Bx)$ is a turbulence producing perturbation. And $F_c(x) = 0.75 \sinh(Bx) \cosh^{-3}(Bx)$ can control this behavior. This can be seen in Fig. 16.

Similarly, the turbulence created by nonlinear damping can be controlled with a stabilizing perturbation:

$$\phi_{tt} - \phi_{xx} - b\phi_t + a\phi_t^3 - G(\phi) = F_c(x). \tag{15}$$

Figs. 17 and 18 show different examples of the controlled dynamics in Eq. (15) using a stabilizing perturbation.

5. Conclusions

The kink-solitons are examples of a very general phenomenon called topological defects. This set of phenomena includes: topological solitons, vortices and spirals [19–21]. Although these objects can possess different origin and nature in different physical systems, they all possess very similar dynamical properties [19–21].

The breakup of topological defects has been observed in experiments in many systems. We can mention here two examples: cardiac tissue and reaction-diffusion systems [22–31,44–53].

In different experiments, it has been observed that one topological defect can breakup into several topological defects. In particular, the “elementary” breakup, where one topological defect breaks up into three topological defects: one antidefect and two defects (see an example in Fig. 6), has been observed in cardiac tissue [44,45,49].

All the situations discussed in the present paper that lead to very complex spatiotemporal behaviors start with kink-soliton breakups (see Figs. 8–11, 13, and 15).

Experimental studies of ventricular fibrillation have suggested that it is caused by multiple reentrant topological waves [27,52]. In this context, ventricular fibrillation is a turbulence of topological defects or, in some cases, it is a highly nonstationary spatiotemporal dynamics where the topological defects play a crucial role. There is a great interest among scientists [27,52] to find the mechanisms that accounts for this turbulence of topological defects.

The investigation of transitions from the regular motion of one topological defect to spatiotemporal chaotic dynamics remains a challenge in science [31]. It is well-known that the instabilities in the form of defect breakup can lead to defect-mediated turbulence [31,32]. During the defect-mediated turbulence a complex spatiotemporal dynamics is generated where new pairs (defect-antidefect) are created and others annihilate. So far there are only heuristic explanations of the breakup phenomenon [31].

At least, the following two different breakup scenarios are documented in experiments [54–56]. In one case, the breakup (leading to turbulence) occurs when a periodic external forcing is added to the system [54]. In a second case, the topological defects break after a Hopf bifurcation [56].

We believe that the results of the present paper show that very similar phenomena can occur with kink-solitons in Klein–Gordon systems. We have been able to produce defect-mediated turbulence using periodic external forcing and after a Hopf bifurcation generated by nonlinear damping.

Excitable media are physical, chemical or biological systems in which besides energy dissipation, in some disturbed regions, there is energy supply [57,58]. These media can support wave motion without attenuation.

Well-known examples are nervous tissue, heart muscle, retinae, and autocatalytic Belousov–Zhabotinsky reactions [19–25,27–32,44,45,54–58].

In excitable media self-sustained dynamical patterns are possible. In this respect, all the mentioned media and physical systems with nonlinear damping, where self-sustained oscillations can exist, are very similar.

In Eq. (9) (with $R(\phi_t) = -b\phi_t + a\phi_t^3$) the energy supply is described by the term $-b\phi_t$. If coefficient b is very small, the defects are stable. However, if b is larger than a critical value, the kink-soliton breakup occurs. Our conjecture is that, in general in excitable systems, the physical elements that are responsible for the energy supply can also be responsible for some of the defect breakups.

Many of the known results about excitable systems have been obtained by real and numerical experiments. Here we have presented an analytical theory of the breakup of Klein–Gordon solitons.

We believe that our results can shed light on some of the still obscure phenomena that occur in excitable systems.

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