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NONINVERTIBLE TRANSFORMATIONS AND SPATIOTEMPORAL RANDOMNESS

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We generalize the exact solution to the Bernoulli shift map. Under certain conditions, the generalized functions can produce unpredictable dynamics. We use the properties of the generalized functions to show that certain dynamical systems can generate random dynamics. For instance, the chaotic Chua's circuit coupled to a circuit with a noninvertible I–V characteristic can generate unpredictable dynamics. In general, a nonperiodic time-series with truncated exponential behavior can be converted into unpredictable dynamics using noninvertible transformations. Using a new theoretical framework for chaos and randomness, we investigate some classes of coupled map lattices. We show that, in some cases, these systems can produce completely unpredictable dynamics. In a similar fashion, we explain why some well-known spatiotemporal systems have been found to produce very complex dynamics in numerical simulations. We discuss real physical systems that can generate random dynamics.

Keywords: Randomness; coupled map lattices; cellular automata.

1. Introduction

In the last few decades, the strands of chaos theory have spread across all sciences like a fractal tree. Chaos theory and nonlinear dynamics have provided new theoretical tools that allow us to understand the complex behaviors of many physical systems [Lorenz, 1993; Schuster, 1995; Jackson, 1991; Moon, 1991; Strogatz, 1994; Glass, 1998].

Deterministic chaotic behavior often looks erratic and random, like the behavior of a system perturbed by external noise. However, the known chaotic systems are not random: precise knowledge of the initial conditions of the system allows us to predict exactly the future behavior of that system, at least in the short term.

In chaotic systems we can observe the divergence of nearby trajectories [Lorenz, 1993; Schuster, 1995; Jackson, 1991; Moon, 1991; Strogatz, 1994; Glass & Mackey, 1998]. This property represents a difference between complex behavior due to deterministic chaos and that due to true randomness [Lorenz, 1993].

This divergence of nearby trajectories leads to a kind of long-term unpredictability. In the random systems we observe immediate unpredictability. Already the next value is unpredictable.

There are processes, as the breaking sea waves on the shore, that are deterministic although they seem random. The behavior of these processes is determined by precise laws.

According to many definitions of randomness, in a random sequence of values, the next value can be any of the previous values with equal probability [Lorenz, 1993]. An example is the coin tossing experiments. Knowing the result of the last coin tossing realization does not increase our chance to guess the result of the next realization.

According to less strict definitions, in a random sequence, the next value can be any of the possible values even if they possess different probabilities, and even if their probability depends on the previous values [Lorenz, 1993]. In other words, for the next outcome there is always more than one possible value.

On the other hand, in a nonrandom sequence, the next value is always determined by the previous values [Lorenz, 1993; Schuster, 1995; Jackson, 1991; Moon, 1991; Strogatz, 1994; Glass & Mackey, 1998].

Can we explain all the randomness we observe in nature using the known temporal chaotic systems?

Another very active area nowadays in nonlinear dynamics is spatiotemporal chaos.

There are several paradigms and model equations for the studying of spatiotemporal and extended systems [Kaneko, 1985, 1992, 1989; Chaté & Manneville, 1988; Crutchfield & Kaneko, 1988; Mayer-Kress & Kaneko, 1989; Kuramoto, 1984; Politi & Torcini, 1992; Kaneko, 1990a; Hansel & Sompolinsky, 1993; Bauer *et al.*, 1993; Bunimovich, 1995; Grassberger & Scheiber, 1991; Kaneko & Konishi, 1989; Kaneko, 1990b; Chaté, 1995; González *et al.*, 1996; González *et al.*, 1998; Guerrero *et al.*, 1999; Pikovsky & Kurths, 1994; Bohr *et al.*, 2001; Grigoriev, 1997; Grigoriev & Schuster, 1998; Shibata & Kaneko, 1998; Wackerbauer & Showalter, 2003; Willeboordse, 2003; Kaneko & Tsuda, 2003].

Coupled map lattices are among the youngest models of extended dynamical systems.

There is a vast literature dedicated to coupled iterated maps [Kaneko, 1985, 1992, 1989; Chaté & Manneville, 1988; Crutchfield & Kaneko, 1988; Mayer-Kress & Kaneko, 1989; Grigoriev & Schuster, 1998; Shibata & Kaneko, 1998; Kaneko & Tsuda, 2003]. Many important numerical results have been obtained in this area. However, the behavior of such coupled systems is quite complex and by no means fully explored.

Are there fundamental differences between the dynamics generated by temporal and spatiotemporal systems?

Researchers have found [Chaté, 1995] that the usual temporal chaos methods of time-series analysis are doomed when the dimension of the spatiotemporal system becomes large (say larger than 10).

On the other hand, it is generally recognized [Kaneko, 1985, 1992, 1989; Chaté & Manneville, 1988; Crutchfield & Kaneko, 1988; Mayer-Kress & Kaneko, 1989; Kuramoto, 1984; Politi & Torcini, 1992; Kaneko, 1990a; Hansel & Sompolinsky, 1993; Bauer *et al.*, 1993; Bunimovich, 1995; Grassberger & Scheiber, 1991; Kaneko & Konishi, 1989; Kaneko, 1990b; Chaté, 1995; González *et al.*, 1996; González *et al.*, 1998; Guerrero *et al.*, 1999; Pikovsky & Kurths, 1994; Bohr *et al.*, 2001; Grigoriev, 1997; Grigoriev & Schuster, 1998; Shibata & Kaneko, 1998; Wackerbauer & Showalter, 2003; Willeboordse, 2003; Kaneko & Tsuda, 2003] that the dynamics of coupled maps is still far from being understood.

Cellular automata conform another class of dynamical systems that has been studied intensively during the last years as simple models for spatially extended systems. In this case, one replaces the continuous variables at each space-time point by discrete ones [von Neumann & Burks, 1996; Wolfram, 1983, 1984a, 1984b, 1986; Hastings *et al.*, 2003; Israeli & Goldenfeld, 2004].

In spite of their simplicity, automaton models are capable of describing many features of physical processes [Wolfram, 1983, 1984a, 1984b, 1986; Hastings *et al.*, 2003; Israeli & Goldenfeld, 2004].

Most results in the field of spatiotemporal systems have been obtained by numerical simulations [Grigoriev & Schuster, 1998].

In the present paper we will show that there exist dynamical systems that can generate completely unpredictable dynamics in the sense that given any string of generated values, for the next outcome, there is always more than one possible value.

The mechanism responsible for the generation of randomness, in a very general class of models and physical systems, is the presence of noninvertible transformations of time-series that contain (truncated) exponential dynamics or chaotic dynamics.

Using a new theoretical framework for randomness we will investigate some classes of coupled map lattices. We will show, that in some cases, these systems can produce completely unpredictable dynamics.

In a similar fashion, we will explain why some elementary cellular automata with very simple rules have been found to produce very complex dynamics in numerical simulations [Wolfram, 1986].

Some consequences of these results in the study of physical and economic systems are discussed.

Some of the concepts discussed in the present paper about the differences between common chaotic and random systems are inspired in [Brown & Chua, 1996] and [González *et al.*, 2000].

2. Unpredictable Dynamics

We will call a time-series $\{X_n\}$ unpredictable if for any string of $m+1$ numbers $X_0, X_1, X_2, \ldots, X_m$ (m can be as large as we wish), then the next number X_{m+1} can take more than one value.

Let us define the general function

$$X_n = P(\theta T z^n), \tag{1}$$

where $P(t)$ is a periodic function, T is the period of function $P(t)$, θ and z are real numbers. An important example of function $P(t)$ is function $P(t) = t \pmod 1$. Note that this is a periodic function with period $T = 1$: $P(t+1) = (t+1) \pmod 1 = P(t)$.

We will show that the dynamics contained in function (1) is unpredictable.

Let us define the family of sequences

$$X_n^{(k,m)} = P[T(\theta_0 + q^m k) z^n], \tag{2}$$

where $z = p/q$ is a rational number such that p and q are relative primes ($p > q$), k and m are integers. The parameter k distinguishes the different sequences. For all sequences parametrized by k, the first $m+1$ values are the same. This is true because

$$\begin{aligned} X_n^{(k,m)} &= P\left[T\theta_0 \left(\frac{p}{q}\right)^n + Tkp^n q^{(m-n)}\right] \\ &= P\left[T\theta_0 \left(\frac{p}{q}\right)^n\right], \end{aligned} \tag{3}$$

for all $n \le m$. Note that the number $kp^n q^{(m-n)}$ is an integer for $n \le m$.

The interesting conclusion is that the next value

$$X_{m+1}^{(k,m)} = P\left[T\theta_0 \left(\frac{p}{q}\right)^{m+1} + \frac{Tkp^{(m+1)}}{q}\right] \tag{4}$$

is unpredictable. $X_{m+1}^{(k,m)}$ can take q different values. For a generic real z, X_{m+1} can take an infinite number of values.

Let us discuss some properties of the following particular case of function (1):

$$X_n = \theta z^n \pmod 1. \tag{5}$$

For $z = 2$, function (5) is the exact solution to the Bernoulli shift map.

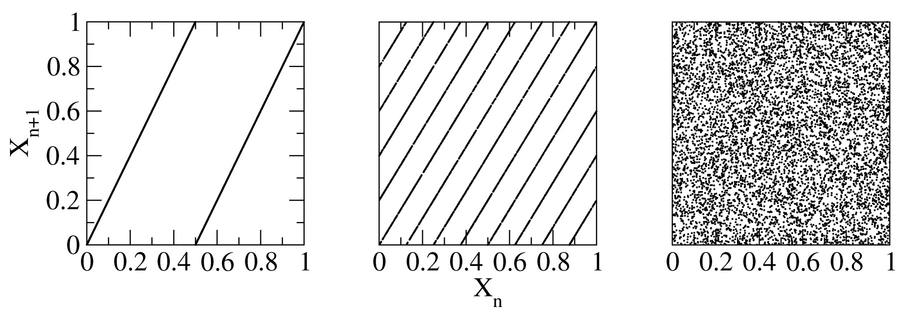
Figures 1(a)–1(c) show different examples of first-return maps produced by the dynamics represented by Eq. (5). The values generated by function (5) are uniformly distributed in the interval $0 < X_n < 1$.

We have generalized these results to functions of type

$$X_n = h[f(n)]. \tag{6}$$

To produce complex dynamics, the function $f(n)$ does not have to be exponential all the time, and function $h(y)$ does not have to be periodic [González *et al.*, 2002]. In fact, it is sufficient for function $f(n)$ to be a finite nonperiodic oscillating function which possesses repeating intervals of truncated exponential behavior. For instance, this can be a common chaotic sequence.

On the other hand, function $h(y)$ should be noninvertible. In other words, it should have different maxima and minima in such a way that equation $h(y) = \alpha$ (for some specific interval of $\alpha, \alpha_1 < \alpha < \alpha_2$) possesses several solutions for y.



(a) (b) (c)

Fig. 1. First-return maps constructed using the dynamics produced by function (5) for $\theta = \pi$. (a) $z = 2$. (b) $z = 8/5$. (c) $z = \pi$.

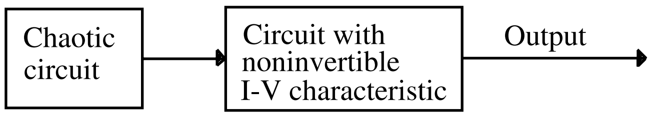


Fig. 2. Scheme of a system that can produce unpredictable dynamics.

Of course, the image of function $f(n)$ should be in the interval where function $h(y)$ is noninvertible.

González *et al.* [2002] have shown that a chaotic Chua's circuit [Matsumoto *et al.*, 1985; Matsumoto *et al.*, 1987] coupled to a Josephson junction can generate unpredictable dynamics. In fact, in order to produce unpredictable dynamics we can use a system with the features shown in Fig. 2.

A method for the construction of circuits with noninvertible *I–V* characteristics can be found in [Chua *et al.*, 1987] and [Comte & Marquié, 2002].

3. Finite Systems of Coupled Maps

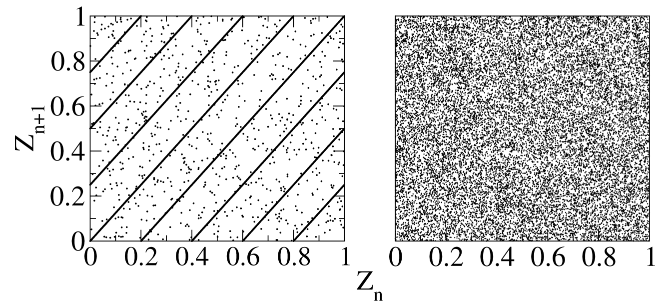
Let us consider the following dynamical system

$$X_{n+1} = \begin{cases} aX_n, & \text{if } X_n < Q, \\ bY_n, & \text{if } X_n > Q, \end{cases} \tag{7}$$

$$Y_{n+1} = cZ_n, \tag{8}$$

$$Z_{n+1} = X_n \pmod 1. \tag{9}$$

Here a can be an irrational number, $a > 1$, $b > 1$, $c > 1$. We can note that for $0 < X_n < Q$, the behavior of function Z_n is exactly like that of function (5). For $X_n > Q$ the dynamics is reinjected to the region $0 < X_n < Q$ with a new initial condition. While X_n is in the interval $0 < X_n < Q$, the dynamics of Z_n is unpredictable as it is function (5). Thus, the process of producing a new initial condition through Eq. (8) is random.



(a) (b)

Fig. 3. First-return maps produced by dynamical system (7)–(9) for $X_0 = Y_0 = Z_0 = 0.1$, $Q = 200$, $b = c = 2$. (a) $a = 5/4$. (b) $a = \pi$.

If the only observable is Z_n, then it is impossible to predict the next values of this sequence using only the knowledge of the past values of $\{Z_n\}$.

An example of the dynamics produced by the dynamical system (7)–(9) is shown in Fig. 3.

In the dynamical system (7)–(9) the variable Z_n is quasi-random, but the variable X_n is predictable because in the interval $0 < X_n < Q$ the rule to determine the next number is a one-valued function.

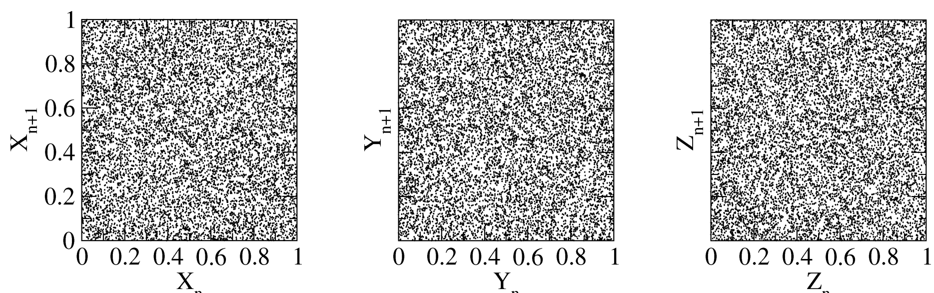
In principle, we can construct dynamical systems where all the variables (taken separately) are random.

Consider the following system:

$$X_{n+1} = \begin{cases} (a + bZ_n)X_n + cY_n, & \text{if } X_n < Q, \\ bY_n, & \text{if } X_n > Q, \end{cases} \tag{10}$$

$$Y_{n+1} = cZ_n, \tag{11}$$

$$Z_{n+1} = X_n \pmod 1. \tag{12}$$



(a) (b) (c)

Fig. 4. Typical dynamics generated by dynamical system (13)–(15). All the variables are unpredictable. Parameter values: $a = 4/3$, $b = 2.1$, $c = 7.3$, $d = 3.1$, $f = 7.7$, $g = 113$. Initial conditions: $X_0 = Y_0 = Z_0 = 0.1$. (a) First-return map of variable X_n. (b) The same for variable Y_n. (c) The same for variable Z_n.

Note that X_n, in Eq. (10), still possesses a finite exponential behavior for $0 < X_n < Q$, because $(a + bZ_n)$ is always a positive number. However, in this case the dynamics of X_n is influenced all the time by the random dynamics of Z_n.

If we are interested in dynamical systems where all the variables are random and uniformly distributed in the interval [0, 1], then we can use the following one:

$$X_{n+1} = [(a + bZ_n)X_n + cY_n + 0.1] \pmod 1, \quad (13)$$

$$Y_{n+1} = [dZ_n + fX_n + 0.1] \pmod 1, \quad (14)$$

$$Z_{n+1} = [gX_n + 0.1] \pmod 1. \quad (15)$$

Here X_n shares many of the properties that are present in the system (10)–(12).

First-return maps of the time-series produced by dynamical system (13)–(15) can be observed in Fig. 4.

4. Coupled Map Lattices

Suppose now that we are interested in symmetric equations in the sense that all equations for X_n, Y_n and Z_n are equivalent.

Note that in the dynamical systems (7)–(9), (10)–(12) and (13)–(15), the equation for Z_{n+1} is constructed in such a way that a nonperiodic dynamics with truncated exponential behavior is the argument of a noninvertible function (say $y = x \pmod 1$). What function is in the argument is not so important. So we can use a function that depends on X_n, but also on Y_n and Z_n as well. On the other hand, the most important feature of the equation for Y_{n+1} is that it depends on the random variable Z_n. So it can also depend on X_n and Y_n. Thus, let us transform dynamical systems (13)–(15) into a symmetric system:

$$X_{n+1} = [(a_1 + b_1Y_n + c_1Z_n)X_n + d_1Y_n + e_1Z_n] \pmod 1, \quad (16)$$

$$Y_{n+1} = [(a_2 + b_2Z_n + c_2X_n)Y_n + d_2Z_n + e_2X_n] \pmod 1, \quad (17)$$

$$Z_{n+1} = [(a_3 + b_3X_n + c_3Y_n)Z_n + d_3X_n + e_3Y_n] \pmod 1. \quad (18)$$

Like the systems discussed before, the set of Eqs. (16)–(18) will produce unpredictable dynamics for all the variables X_n, Y_n and Z_n taken separately. This can be seen in Figs. 5(a)–5(c).

Can we construct a coupled map lattice with these characteristics?

Now we will have a dynamical variable that depends on the time n and the space coordinate i. Instead of three equations with three variables as in system (16)–(18), we will have an infinite number of equations. Our variable will be $X_n(i)$.

An example of a coupled map lattice with all the properties discussed above is the following:

$$X_{n+1}(i) = [(a + bX_n(i-1) + cX_n(i+1))X_n(i) + dX_n(i-1) + fX_n(i+1) + 0.1] \pmod 1. \quad (19)$$

Note that for each space site i, we have a nonperiodic dynamics with truncated exponential behavior that depends on the behavior of the space sites $(i - 1)$ and $(i + 1)$. This dynamics is always the argument of a noninvertible function (in this case $y = x \pmod 1$). We are sure that the dynamics is nonperiodic because even something as simple as $X_{n+1}(i) = aX_n(i) \pmod 1$ would produce chaotic behavior for $a > 1$.

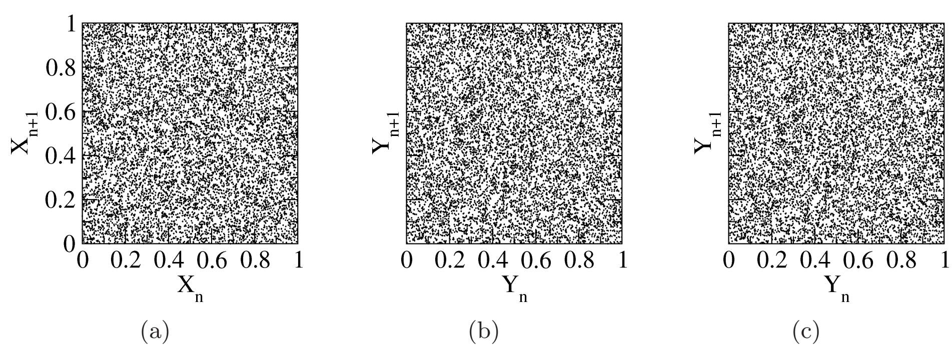
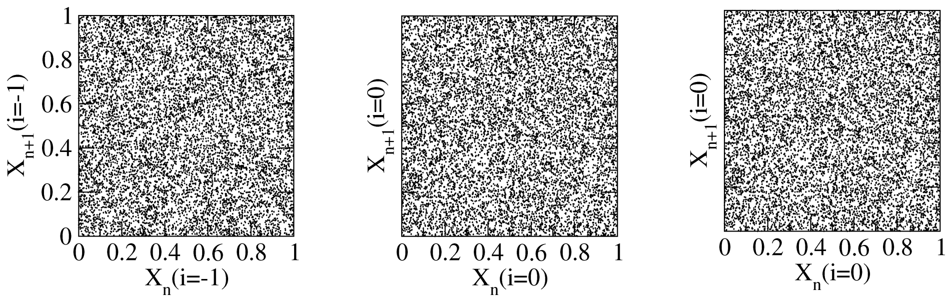


Fig. 5. Dynamics produced by the set of Eqs. (16)–(18). Parameter values: $a_1 = 1.3$, $b_1 = \pi$, $c_1 = 2.6$, $d_1 = 1.5$, $e_1 = 1.1$, $a_2 = 4.6$, $b_2 = 2.1$, $c_2 = e$, $d_2 = 3.2$, $e_2 = 7.1$, $a_3 = 2.9$, $b_3 = 5.4$, $c_3 = 8.7$, $d_3 = 4.5$, $e_3 = 1.9$. Initial conditions: $X_0 = Y_0 = Z_0 = 0.1$. (a) First-return map of variable X_n. (b) The same for variable Y_n. (c) The same for variable Z_n.



(a) (b) (c)

Fig. 6. Typical dynamics generated by the coupled map lattice defined by Eq. (19). Parameter values: $a = 2$, $b = c = d = f = 1$. Initial condition: $X_0 = 0.1$. (a) $i = -1$. (b) $i = 0$. (c) $i = 1$.

Another interesting example of coupled map lattices with random behavior can be found in the system

$$X_{n+1}(i) = [(a + bX_n(i-1) + cX_n(i) + dX_n(i-1))X_n(i) + fX_n(i-1) + gX_n(i+1) + 0.1] \pmod 1. \qquad (20)$$

Here the coefficient of variable $X_n(i)$ in the argument of the modulo function depends on $X_n(i-1)$, $X_n(i+1)$ and the same $X_n(i)$.

Figures 6(a)–6(c) show the dynamics generated by different sites in the introduced coupled map lattices.

5. Cellular Automata

The values of the sites in a one-dimensional cellular automaton are updated in parallel in discrete time steps according to a rule of the form

$$Y_{n+1}(i) = F[Y_n(i-r), Y_n(i-r+1), \ldots, Y_n(i+r)]. \qquad (21)$$

The site values are usually taken as integers between zero and $(k-1)$ [Wolfram, 1983, 1984a, 1984b, 1986].

Cellular automata can be considered as discrete approximations to partial differential equations, and used as direct models for a wide class of natural systems [Wolfram, 1983, 1984a, 1984b, 1986; Hastings *et al.*, 2003; Israeli & Goldenfeld, 2004].

A classification and several studies of the cellular automata with $k = 2$ and $r = 1$ can be found in [Wolfram, 1983, 1984a, 1984b, 1986].

Representations of the so-called Rules 30, 110 and 124 are shown in Tables 1–3.

The top row in each set of three elements gives one of the possible combinations of values for a cell and its immediate neighbors. The bottom row then specifies what value the center cell should have on the next step in each of these cases.

Rules 110 and 124 are equivalent under reflection transformations [Wolfram, 1983].

Rules 110 and 124 are relevant because they have been proved to be equivalent to Turing machines. So they are capable of universal computations [Israeli & Goldenfeld, 2004].

On the other hand, Rule 30 has been considered as a model of randomness in nature and has been used as a practical pseudorandom number generator [Wolfram, 1986].

Rule 30 can be written as a coupled map lattice:

$$Y_{n+1}(i) = [Y_n(i-1) + Y_n(i) + Y_n(i+1) + Y_n(i)Y_n(i+1)] \pmod 2. \qquad (22)$$

The sequences generated by Rule 30 have been analyzed by a variety of empirical and statistical

Table 1. Representation of Rule 30 cellular automaton.

111	110	101	100	011	010	001	000
0	0	0	1	1	1	1	0

Table 2. Representation of Rule 110 cellular automaton.

111	110	101	100	011	010	001	000
0	1	1	0	1	1	1	0

Table 3. Representation of Rule 124 cellular automaton.

111	110	101	100	011	010	001	000
0	1	1	1	1	1	0	0

techniques [Wolfram, 1986] and the researchers have concluded that they seem completely random.

Random sequences are obtained from Rule 30 by sampling the values that a particular site attains as a function of time.

An example very frequently used is the apparent "randomness" of the center vertical column in the patterns shown in Fig. 7. The evolution of Rule 124 cellular automaton is shown in Fig. 8.

In all these works the authors recognize that little has been proved theoretically about Rule 30. However, the center vertical sequence has passed all the statistical tests of randomness applied to it.

If a point that separates the integer part from the fractionary part is placed near the central column as is shown in Figs. 9 and 10, then the outcomes of the cellular automaton evolution can be transformed into a numerical time series $\{Y_n\}$, where the Y_n are real numbers written in binary system.

In the case of Rule 124 this is always a bounded time series where $0 \le Y_n \le 1$.

The first-return map of a typical Rule 124 time series is shown in Fig. 11. Note that the function $Y_{n+1} = f(Y_n)$ is a fractal. Nevertheless, it is important to notice that despite its fractal structure this

```
                                      1
                                    1 1 1
                                  1 1 0 0 1
                                1 1 0 1 1 1 1
                              1 1 0 0 1 0 0 0 1
                            1 1 0 1 1 1 1 0 1 1 1
                          1 1 0 0 1 0 0 0 0 1 0 0 1
                        1 1 0 1 1 1 1 0 0 1 1 1 1 1 1
                      1 1 0 0 1 0 0 0 1 1 1 0 0 0 0 0 1
                    1 1 0 1 1 1 1 0 1 1 0 0 1 0 0 0 1 1 1
                  1 1 0 0 1 0 0 0 0 1 0 1 1 1 1 0 1 1 0 0 1
                1 1 0 1 1 1 1 0 0 1 1 0 1 0 0 0 0 1 0 1 1 1 1
              1 1 0 0 1 0 0 0 1 1 1 0 0 1 1 0 0 1 1 0 1 0 0 0 1
            1 1 0 1 1 1 1 0 1 1 0 0 1 1 1 0 1 1 1 0 0 1 1 0 1 1 1
          1 1 0 0 1 0 0 0 0 1 0 1 1 1 0 0 0 1 0 0 1 1 1 0 0 1 0 0 1
        1 1 0 1 1 1 1 0 0 1 1 0 1 0 0 1 0 1 1 1 1 1 0 0 1 1 1 1 1 1 1
      1 1 0 0 1 0 0 0 1 1 1 0 0 1 1 1 1 0 1 0 0 0 0 1 1 1 0 0 0 0 0 0 1
    1 1 0 1 1 1 1 0 1 1 0 0 1 1 1 0 0 0 0 1 1 0 0 1 1 0 0 1 0 0 0 0 1 1 1
  1 1 0 0 1 0 0 0 0 1 0 1 1 1 0 0 1 0 0 1 1 0 1 1 1 0 1 1 1 1 0 0 1 1 0 0 1
1 1 0 1 1 1 1 0 0 1 1 0 1 0 0 1 1 1 1 1 1 0 0 1 0 0 0 1 0 0 0 1 1 1 0 1 1 1 1
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Fig. 7. Evolution of Rule 30 cellular automaton.

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                                      0 . 1
                                    1 . 1 1
                                  1 1 . 0 0 1
                                1 1 0 . 1 1 1 1
                              1 1 0 0 . 1 0 0 0 1
                            1 1 0 1 1 . 1 1 0 1 1 1
                          1 1 0 0 1 0 . 0 0 0 1 0 0 1
                        1 1 0 1 1 1 1 . 0 0 1 1 1 1 1 1
                      1 1 0 0 1 0 0 0 . 1 1 1 0 0 0 0 0 1
                    1 1 0 1 1 1 1 0 1 . 1 0 0 1 0 0 0 1 1 1
                  1 1 0 0 1 0 0 0 0 1 . 0 1 1 1 1 0 1 1 0 0 1
                1 1 0 1 1 1 1 0 0 1 1 . 0 1 0 0 0 0 1 0 1 1 1 1
              1 1 0 0 1 0 0 0 1 1 1 0 . 0 1 1 0 0 1 1 0 1 0 0 0 1
            1 1 0 1 1 1 1 0 1 1 0 0 1 . 1 1 0 1 1 1 0 0 1 1 0 1 1 1
          1 1 0 0 1 0 0 0 0 1 0 1 1 1 . 0 0 0 1 0 0 1 1 1 0 0 1 0 0 1
        1 1 0 1 1 1 1 0 0 1 1 0 1 0 0 . 1 0 1 1 1 1 1 0 0 1 1 1 1 1 1 1
      1 1 0 0 1 0 0 0 1 1 1 0 0 1 1 1 . 1 0 1 0 0 0 0 1 1 1 0 0 0 0 0 0 1
    1 1 0 1 1 1 1 0 1 1 0 0 1 1 1 0 0 . 0 0 1 1 0 0 1 1 0 0 1 0 0 0 0 1 1 1
  1 1 0 0 1 0 0 0 0 1 0 1 1 1 0 0 1 0 . 0 1 1 0 1 1 1 0 1 1 1 1 0 0 1 1 0 0 1
1 1 0 1 1 1 1 0 0 1 1 0 1 0 0 1 1 1 1 . 1 1 0 0 1 0 0 0 1 0 0 0 1 1 1 0 1 1 1 1
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Fig. 9. The outcomes of Rule 30 can be seen as a numerical time series of real numbers written in binary representation.

```
                                      1
                                    0 1 1
                                  0 0 1 1 1
                                0 0 0 1 0 1 1
                              0 0 0 0 1 1 1 1 1
                            0 0 0 0 0 1 0 0 0 1 1
                          0 0 0 0 0 0 1 1 0 0 1 1 1
                        0 0 0 0 0 0 0 1 1 1 0 1 0 1 1
                      0 0 0 0 0 0 0 0 1 0 1 1 1 1 1 1 1
                    0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 0 0 1 1
                  0 0 0 0 0 0 0 0 0 0 1 0 1 1 0 0 0 0 1 1 1
                0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0 0 0 1 0 1 1
              0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 1 0 0 1 1 1 1 1
            0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 1 0 1 0 0 0 1 1
          0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 0 1 1 1 1 0 0 1 1 1
        0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 1 1 0 0 1 1 0 1 0 1 1
      0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 1 0 1 1 1 1 1 1 1 1
    0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 0 0 1 1 1 1 0 0 0 0 0 0 1 1
  0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0 1 0 0 1 1 0 0 0 0 0 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 1 1 1 0 1 1 1 0 0 0 0 1 0 1 1
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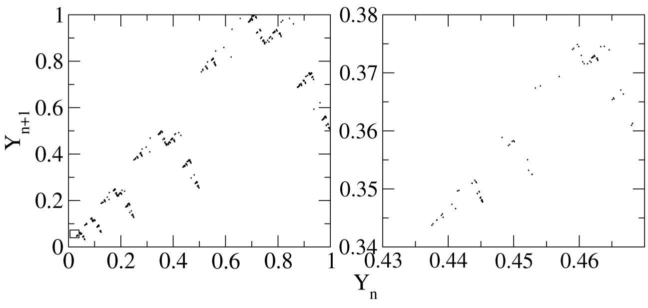
Fig. 8. Evolution of Rule 124 cellular automaton.

```
                                      0. 1
                                    0 0. 1 1
                                  0 0 0. 1 1 1
                                0 0 0 0. 1 0 1 1
                              0 0 0 0 0. 1 1 1 1 1
                            0 0 0 0 0 0. 1 0 0 0 1 1
                          0 0 0 0 0 0 0. 1 1 0 0 1 1 1
                        0 0 0 0 0 0 0 0. 1 1 1 0 1 0 1 1
                      0 0 0 0 0 0 0 0 0. 1 0 1 1 1 1 1 1 1
                    0 0 0 0 0 0 0 0 0 0. 1 1 1 0 0 0 0 0 1 1
                  0 0 0 0 0 0 0 0 0 0 0. 1 0 1 1 0 0 0 0 1 1 1
                0 0 0 0 0 0 0 0 0 0 0 0. 1 1 1 1 1 0 0 0 1 0 1 1
              0 0 0 0 0 0 0 0 0 0 0 0 0. 1 0 0 0 1 1 0 0 1 1 1 1 1
            0 0 0 0 0 0 0 0 0 0 0 0 0 0. 1 1 0 0 1 1 1 0 1 0 0 0 1 1
          0 0 0 0 0 0 0 0 0 0 0 0 0 0 0. 1 1 1 0 1 0 1 1 1 1 0 0 1 1 1
        0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0. 1 0 1 1 1 1 1 0 0 1 1 0 1 0 1 1
      0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0. 1 1 1 0 0 0 1 1 0 1 1 1 1 1 1 1 1
    0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0. 1 0 1 1 0 0 1 1 1 1 0 0 0 0 0 0 1 1
  0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0. 1 1 1 1 1 0 1 0 0 1 1 0 0 0 0 0 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0. 1 0 0 0 1 1 1 1 0 1 1 1 0 0 0 0 1 0 1 1
```

Fig. 10. Binary representation of the time series produced by Rule 124.



(a) (b)

Fig. 11. First-return map constructed using the sequence $\{Y_n\}$ produced with the dynamics of Rule 124 cellular automaton as described in the main text. (a) Full first-return map. (b) Zoom of a detail of the first-return map.

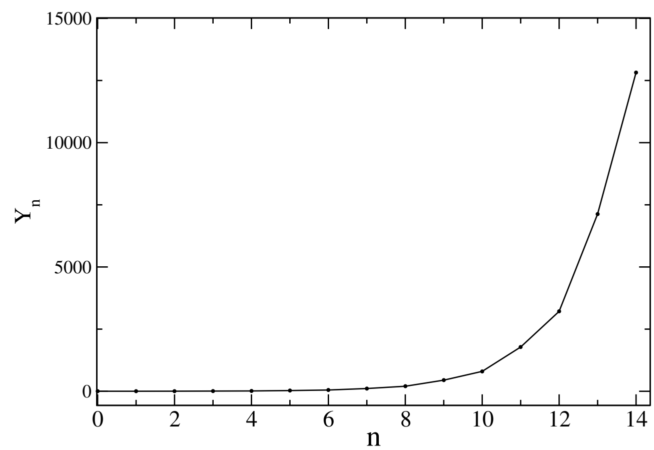


Fig. 12. Approximate exponential behavior of $\{Y_n\}$ for Rule 30.

is a one-valued first-return map. In fact, given any previous value Y_n, the next value is always defined by this previous value. We have calculated numerically the Lyapunov exponent of this map using different methods of time-series analysis, see e.g. [Wolf *et al.*, 1985] and [Kantz & Schreiber, 1997]. In these calculations, we generate the sequence $\{Y_n\}$ using the cellular automaton rule. Then we treat the produced sequence as an experimental time-series, and we compute the largest Lyapunov exponent using the mentioned standard methods. In our calculations, the largest Lyapunov exponent is approximately $\lambda = 0.4$. All this leads to the speculation that a dynamical system, that can be mapped to a fractal chaotic map of type $Y_{n+1} = f(Y_n)$, is capable of universal computation.

The geometrical structure shown in Fig. 11 is an invariant and can be used to have a general representation of the dynamics for any initial condition. It is independent of time.

We believe that this kind of representation is more general and useful that the Wolfram's "space-time" calculations of several hundreds or thousands steps, because they are by definition limited and misleading. The dynamics that can be observed in an interval of time of 1,000 steps can be very different in an interval of time taken 1,000,000 steps away.

On the other hand, for Rule 30, the time series $\{Y_n\}$ is an unbounded exponentially increasing function (see Fig. 12). In fact, $\{Y_n\}$ can be expressed as a map of type

$$Y_{n+1} = a_n Y_n, \tag{23}$$

where a_n always takes noninteger values such that $a_n > 1$.

From the representation of Rule 30 in Table 1 it is evident that from a number

$$Y_n = \cdots b_{-3}b_{-2}b_{-1} \cdot b_1 b_2 b_3 b_4 \cdots \tag{24}$$

where b_k, b_{-k} are zeroes or ones; the number

$$Y_{n+1} = \cdots b'_{-3}b'_{-2}b'_{-1} \cdot b'_1 b'_2 b'_3 b'_4 \cdots \tag{25}$$

can be obtained only using a noninteger a_n which should be close to 2 (see the actual evolution in Fig. 9). The behavior of a_n can be seen in Fig. 13.

In fact, a numerical calculation shows that the dynamics of $\{a_n\}$ possesses a quasiperiodic attractor (see Fig. 13) where all the values of a_n are close to two possible values: 1.8 and 2.2.

Thus, Y_n is approximately an exponentially increasing function. All we need to produce

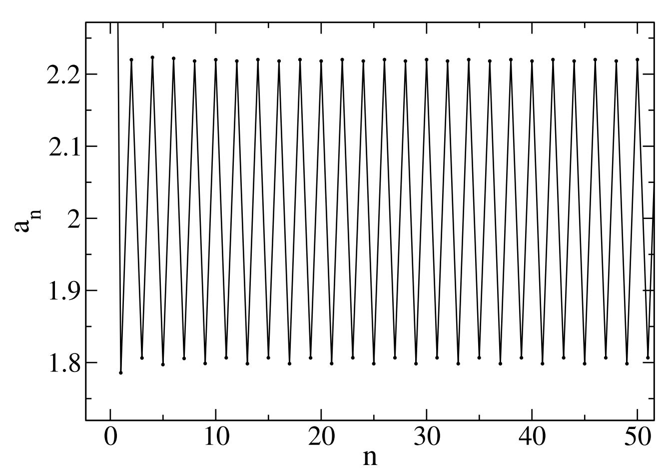


Fig. 13. Behavior of a_n as defined in Eq. (23).

unpredictable dynamics is the application of a noninvertible transformation on Y_n. For instance, the function

$$X_n = Y_n \pmod 1 \tag{26}$$

is much harder to predict than Rule 124. The first-return map of this dynamics can be observed in Fig. 14. Note that the first-return map is two-valued. Given a X_n, we always have two possible future values X_{n+1}.

Notice that if $a_n = 2$, the time-series is predictable as in the Bernoulli shift.

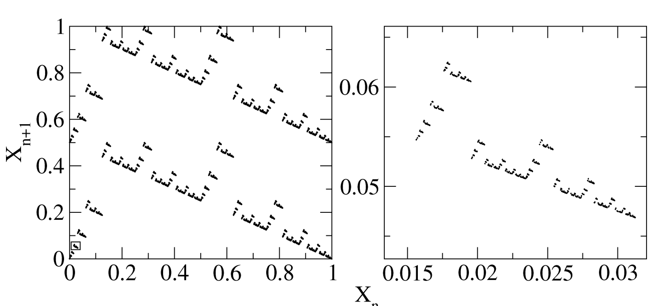
So it is important that, due to the structure of Rule 30, the resulting a_n are always noninteger. This leads to unpredictability.

Other noninvertible transformations can be used to produce unpredictability from Rule 30.

The operation of sampling the value that a particular site attains as a function of time is a noninvertible transformation. This can be observed in Table 1, where Rule 30 is represented. So, by sampling the value of the central column in the Rule 30 evolution (see Fig. 9), which is equivalent to a sequence with exponential behavior, we are generating a much more unpredictable dynamics than that produced by Rule 124.

All the studies about this cellular automaton conclude that its dynamics is computationally as sophisticated as any physically realizable system can be [Wolfram, 1983, 1984a, 1984b, 1986].

The authors of these works say that it is computationally irreducible and its outcome can effectively be found only by direct simulation or observation. So there should be no general computational shortcuts or finite mathematical procedure to investigate its behavior. As a consequence, all the questions concerning infinite time or infinite size limits cannot be answered by bounded computations.



(a) (b)

Fig. 14. First-return map produced by Eq. (26), where Y_n is the time-series generated by Rule 30 cellular automaton. (a) Full map. (b) Detail.

In fact, they cannot be sure if after a very large number of time steps, the dynamics generated by Rule 30 can become periodic.

However, following our results, we can predict that the dynamics produced by sampling the values that a particular site attains as a function of time is nonperiodic.

From this, we arrive at a very important conclusion concerning forecasting methods in distributed systems. Although the general spatiotemporal dynamics can be deterministic, the local dynamics can seem completely random.

In other words, chaotic spatiotemporal systems can produce completely random dynamics locally. However, a knowledge of system spatiotemporal dynamics can lead to correct predictions, at least in the short term.

6. Other Noninvertible Transformations

The operation of calculating the mean value of several time-series is a noninvertible transformation.

Usually it is assumed that the average value of a quantity will be a more simple dynamics than the dynamics of the quantity itself.

Let us discuss the situation represented in Table 4. The values of each column are produced using the chaotic map

$$X_{n+1} = 5.3X_n \pmod 1, \tag{27}$$

but with different initial conditions.

The dynamics in each column is chaotic but predictable. This can be seen in the first-return map shown in Fig. 15(a). Given a X_n, the next value is uniquely determined.

Now let us define a new variable Y_n as the mean value of the values X_n that appear in each row of Table 4. The result is a time series whose complexity depends on the number of averaged columns [see Figs. 15(b)–15(d)]. N is the number of averaged column values.

Note that for $N \to \infty$, the distribution of Y_n tends to be Gaussian as expected from the Central Limit Theorem. When the chaotic map used for generating the columns is the logistic, the dynamics of Y_n is shown in Fig. 16.

It is important to remark that different forecasting methods also corroborate that the dynamics of Y_n becomes unpredictable [Farmer & Sidorowich, 1987; Sugihara & May, 1990].

Table 4. The columns that correspond to X_n are produced using the map (27) with different initial conditions. The variable Y_n is the average value of the different X_n for a given n.

n	Y_n	X_n									
0	0.1450	0.1000	0.1100	0.1200	0.1300	0.1400	0.1500	0.1600	0.1700	0.1800	0.1900
1	0.6685	0.5300	0.5830	0.6360	0.6890	0.7420	0.7950	0.8480	0.9010	0.9540	0.0070
2	0.4430	0.8090	0.0899	0.3708	0.6517	0.9326	0.2135	0.4944	0.7753	0.0562	0.0371
3	0.4482	0.2877	0.4765	0.9652	0.4540	0.9428	0.1316	0.6203	0.1091	0.2979	0.1966
4	0.4753	0.5248	0.5253	0.1158	0.4063	0.9967	0.6972	0.2877	0.5782	0.5787	0.0421
5	0.4190	0.7815	0.7840	0.6136	0.1532	0.2827	0.6953	0.5248	0.0644	0.0669	0.2233
6	0.4204	0.1418	0.1554	0.2519	0.8117	0.4982	0.6848	0.7813	0.3411	0.3545	0.1837
7	0.6284	0.7515	0.8239	0.3352	0.3021	0.6404	0.6296	0.1411	0.8079	0.8788	0.9735
8	0.5304	0.9827	0.3665	0.7767	0.6010	0.3939	0.3370	0.7476	0.2817	0.6575	0.1598
9	0.5112	0.2084	0.9422	0.1164	0.1850	0.0875	0.7859	0.9621	0.4930	0.4846	0.8468

The average operation can produce complexity also when we have only one time-series.

Suppose X_n is a time-series produced by the chaotic map (27). Now define Y_k as:

$$Y_k = \frac{1}{N} \sum_{n=k}^{N+k} X_n. \tag{28}$$

This dynamics will have the same properties as that obtained by the averaging of several different chaotic time-series.

These results could be relevant to investigations of thermostatistical and economic systems where averaging of chaotic quantities is a common practice.

The "randomness" found in the so-called Bernoulli random variables [Denker & Woyczynski, 1998] is also the result of the application of a non-invertible transformation to a chaotic time-series.

This phenomenon can be seen in the following example:

$$Y_n = \phi(X_n), \tag{29}$$

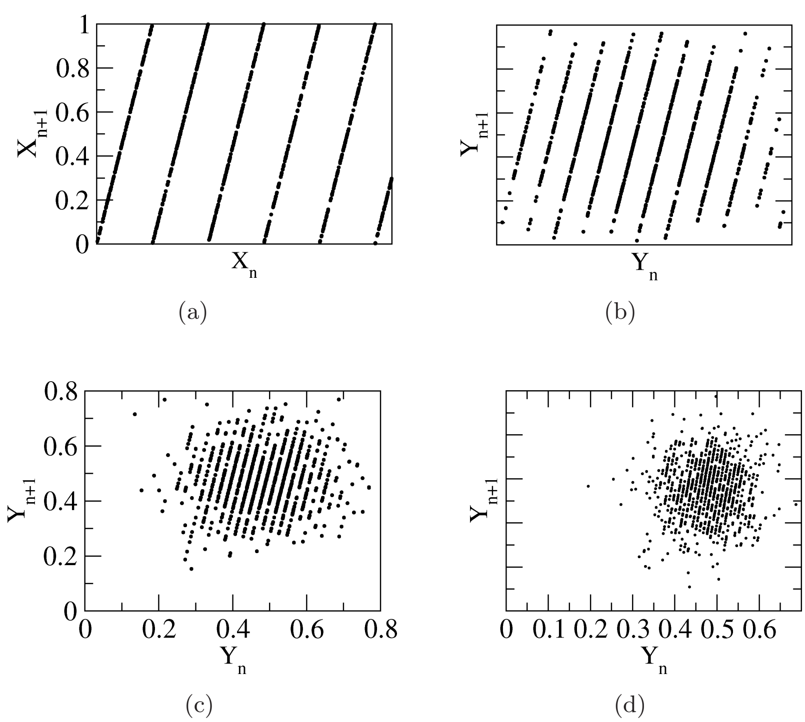


Fig. 15. First return maps for X_n as defined in Eq. (27) and the variable Y_n, which is the average value of the different X_n as explained in Table 4. (a) $N = 1$. (b) $N = 2$. (c) $N = 8$. (d) $N = 20$.

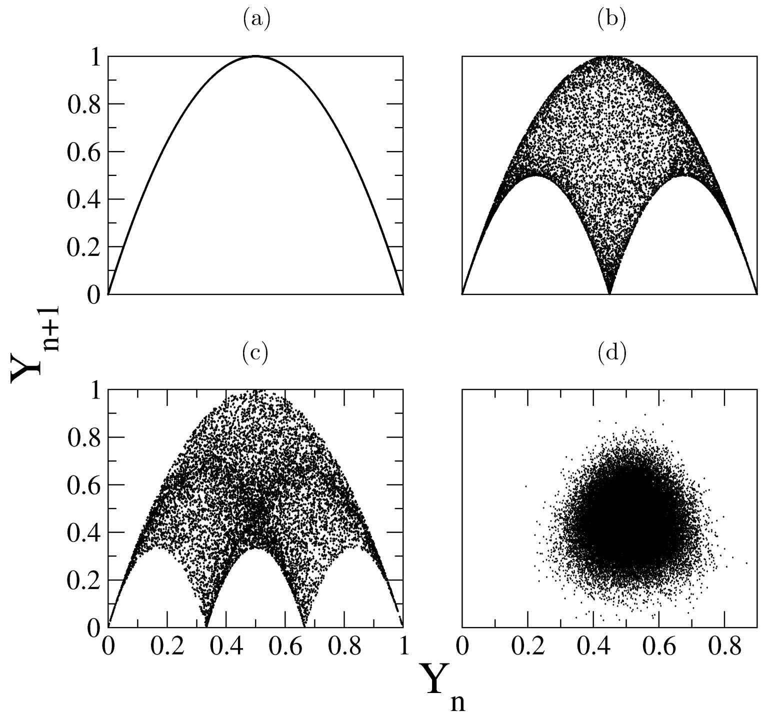


Fig. 16. The same as in Fig. 15 but the generating dynamical system is the logistic map. (a) $N = 1$. (b) $N = 2$. (c) $N = 3$. (d) $N = 10$.

where $X_{n+1} = aX_n \pmod 1$, a is a noninteger number, $a > 1$, and $\phi(t)$ is defined as follows:

$$\phi(t) = \begin{cases} 1, & \text{if } t \geq \dfrac{1}{2}, \\ 0, & \text{if } t < \dfrac{1}{2}. \end{cases} \tag{30}$$

Note that $\phi(t)$ is a noninvertible function.

A statistical investigation of the time-series produced by Eq. (29) will show that it has the same properties as the Rule 30 central column time-series.

7. Conclusions

We have shown that there exist dynamical systems that can generate completely unpredictable dynamics in the sense that, given any string of generated values, for the next outcome, there is always more than one possible value.

The mechanism responsible for the generation of randomness, in a very general class of models and physical systems, is the presence of noninvertible transformations of time-series that contain nonperiodic (truncated) exponential dynamics or chaotic behavior.

Using a new theoretical framework for randomness, we have investigated some classes of coupled map lattices. We have shown that, in some cases, these systems can produce completely unpredictable dynamics.

Spatiotemporally chaotic systems can produce locally unpredictable dynamics even when the global spatiotemporal dynamics is completely deterministic. An example can be an array of coupled Josephson junctions perturbed by a chaotic circuit like Chua's circuit.

Local measurements of a quantity that characterizes a phenomenon in a complex system (like the climate or the seismic events) can generate completely unpredictable time-series. However, the global spatiotemporal data of the phenomenon can provide the necessary information for accurate predictions, at least in the short term.

When dealing with spatiotemporal complexity, a necessary step is an investigation of the full space-time dynamics [Chaté, 1995]. Local probes alone are not sufficient for efficient predictions.

Researchers need both the local and the complete spatiotemporal dynamics to reveal the important features. The measurements should be made in an extended zone. The more extended the zone, the better. An experimental setup should allow the acquisition of the full space-time information.

These results are also important in the study of economic systems where noninvertible operations are a common practice.

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