In: Handbook of Solitons
Editors: S.P. Lang and S.H. Bedore, pp. 587-620
ISBN 978-1-60692-596-6

Chapter 16

Control of Chaotic and Exploding Dissipative Kink-Solitons

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Abstract

Dissipative kink-solitons, from non complete integrable systems, exhibit a richer internal structure than its non-dissipative counterparts. Herein, we present an analytical investigation on what makes the existence of this internal structure possible (giving along the way proof on the existence of internal modes in the perturbed sine-Gordon systems), and how these internal modes play a role in the kink-soliton stability. We will show how the dissipative kink-soliton behaves as an extended (nonrigid) object, where phenomena such as kink-soliton explosion, and chaotic dynamics induced by the internal modes are possible. Furthermore, we present ways in which such situations can be avoided, either through direct access to the parameters, or by applying a generalized concept of geometrical resonance when access to the parameters is not possible. We will show how, by means of the generalized concept of geometrical resonance, different dynamical patterns can be stabilized and avoid chaotic dynamics (for instance, the stabilization of large amplitude breathers and oscillatory patters avoiding chaos). Moreover, this method can be used to suppress spatiotemporal chaos and turbulence in systems where these phenomena are already present by means of applying simple localized external forcing. These methods can be generalized to even more general spatiotemporal systems.

1. Introduction

The topological solitons studied here possess important applications in condensed matter physics – they describe domain walls in ferromagnets and ferroelectric materials, dislocations in crystals, charge-density waves, interphase boundaries in metal alloys, fluxons in long Josephson junctions and Josephson transmission lines, nonlinear magnetization waves, etc. [1–11]

Additionally, solitons are used in many important technological applications. Among these applications we can mention long distance communication systems, soliton oscillators in superconducting devices [12], and soliton transistors [13].

In this chapter we will investigate dissipative kink-solitons, from non complete integrable systems, as they exhibit a richer internal structure than its non-dissipative counterparts. This internal structure can lead to spatiotemporal chaos [14–18].

Spatiotemporal chaos is one of the most important (and most studied) phenomena of recent years. Chaos can be advantageous in some situations, while in many other situations, it should be avoided or controlled [19–27]. In certain cases, the desired effect is a high-amplitude periodic oscillation. We should drive a nonlinear system with a large external force to produce such a high-amplitude oscillation. However, this should be done in such a way that chaos is avoided. Different feedback mechanisms have been devised to control chaos [20; 28–30]. A great deal of research has been dedicated also to the problem of suppresing chaos by harmonic (or just periodic) perturbations [24; 27; 31–38]. Among those works are the ones that use the concept of Geometrical Resonance (GR) [27; 34; 36–40].

This chapter is organized as following: In Section 2. we present the general conditions for kink-soliton existence and stability. Section 3. deals with the existence of kink-soliton internal modes, specifically the case of sine-Gordon solitons. There, using the results obtained in Sec. 2. we find conditions for the internal modes to exist. Section 4. is dedicated to the phenomena of kink-soliton explosion, there we study different mechanisms that lead to the kink-soliton breakup. In Section 5. we introduce the general concept of Geometrical Resonance for spatiotemporal systems, and using the GR condition we show how different patterns can be stabilized. Finally, in Section 6. we discuss some general conclusions and experimental considerations.

2. Kink-soliton Existence and Stability

In this section we will be dealing with the existence and stability of kink-solitons from systems described by the following equation

$$\phi_{tt} + \mathrm{R}\left(\phi_t\right) - \phi_{xx} - \mathrm{G}\left(\phi\right) = \mathrm{F}(x,t), \tag{1}$$

where ϕ_{tt} and ϕ_{xx} represent the partial derivatives with respect to time and space respectively, $\mathrm{R}\left(\phi_t\right)$ are dissipative terms (they may be linear or not), $\mathrm{G}\left(\phi\right) = -d\mathrm{U}(\phi)/d\phi$, $\mathrm{U}(\phi)$ is a potential function, and $\mathrm{F}(x,t)$ represents external perturbations.

2.1. Kink-soliton Existence

It is well known that static kink-solitons are the solution on the heteroclinic trajectories joining the fixed points of the corresponding dynamical system [41] (see Fig. 1). This implies that the effective potential of eq. (1) must at least possess three extrema points: two minima at ϕ_1 and ϕ_3, and a maximum at ϕ_2, such that $\mathrm{U}(\phi_1) = \mathrm{U}(\phi_3)$[18].

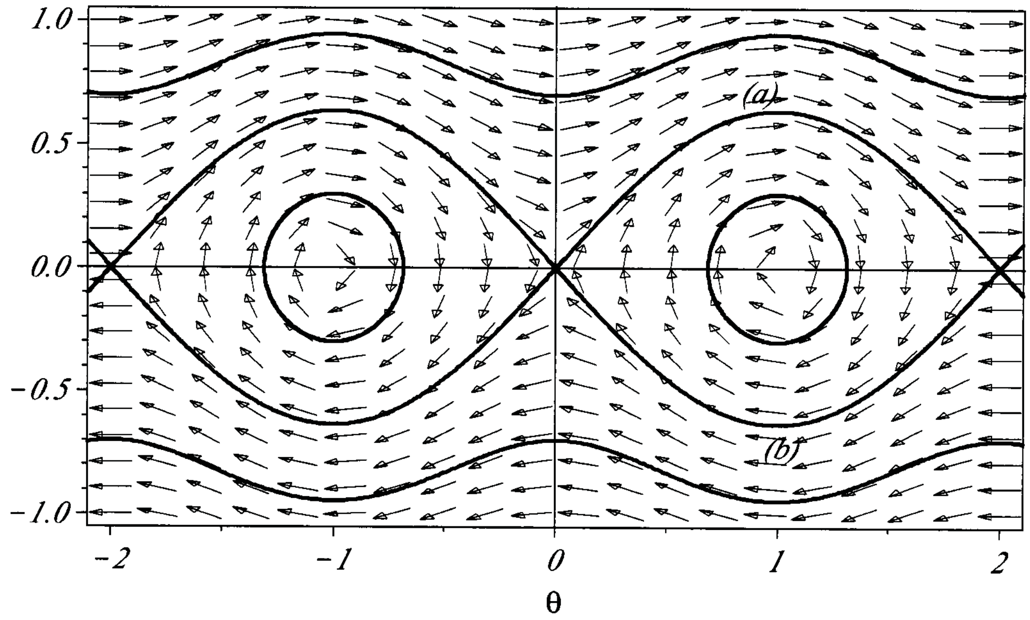

Figure 1. Heteroclinic trajectories. Upper (a) and lower (b) heteroclinic trajectories represents the static kink-soliton, and static antikink-soliton, respectively.

With this condition in mind we can find (and define) the strength of a continuous external force needed to destroy the kink-soliton.

Assume for simplicity that the external perturbation is constant ($\mathrm{F}(x,t) = \mathrm{F}$), and there are not dissipative terms ($\mathrm{R}\,(\phi_t) = 0$)

$$\phi_{tt} - \phi_{xx} - \mathrm{G}\,(\phi) = \mathrm{F}, \tag{2}$$

where we can define an effective potential $\mathrm{U}_{Eff}(\phi) = \mathrm{U}(\phi) - \mathrm{F}\phi$, and find the smallest value of F that will destroy the heteroclinic trajectories. Let us call this value F_c, the value for which the heteroclinic trajectories cease to exist.

It can be shown that this is also true for the case where linear dissipation is considered ($\mathrm{R}\,(\phi_t) = \gamma\phi_t$).

In the case we consider an inhomogeneous space-dependent perturbation ($\mathrm{F}(x,t) = \mathrm{F}(x)$).

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} - \mathrm{G}\,(\phi) = \mathrm{F}(x). \tag{3}$$

It is known that the zeroes of $\mathrm{F}(x)$ are candidates for equilibrium positions of the kink-soliton [18; 42; 43]; and if F(x) possesses at least one zero at x_0^* ($\mathrm{F}(x_0^*) = 0$), then it will be a stable position for the kink-soliton if $d\mathrm{F}(x)/dx]_{x=x_0^*} > 0$ and otherwise for the antikink-soliton. However, if it is not true (x_0^* is an unstable position), we need to consider the strength of the inhomogeneous perturbation at the extremes, $\lim_{x\to\pm\infty} \mathrm{F}(x) = \mp\mathrm{F}_\infty$, where $\mathrm{F}_\infty > 0$. As before, we need this value (F_∞) to be less than F_c or the kink-solitons will be destroyed.

Let us now see what would happen when nonlinear dissipation is considered:

$$\phi_{tt} + \mathrm{R}(\phi_t) - \phi_{xx} - \mathrm{G}(\phi) = 0, \tag{4}$$

where $d\mathrm{R}(\phi_t)/d\phi_t$ is negative for small values of $|\phi_t|$, and positive elsewhere.

Since we are interested in solutions of the form $\phi(x,t) = \phi(x - vt)$, where v is the kink-soliton velocity, then rewriting (4) as an autonomous dynamical system:

$$\begin{aligned} \phi_z &= \theta, \\ \theta_z &= \mathrm{R}(-w\theta) - \mathrm{G}(\phi), \end{aligned} \tag{5}$$

where $z = (x - vt)/\sqrt{1 - v^2}$ and $w = v/\sqrt{1 - v^2}$; and the heteroclinic trajectories correspond to stationary kink-solitons solutions of (4).

In the case that $\mathrm{R}(0) = 0$, then the points $(\phi_1, 0)$ and $(\phi_3, 0)$ are saddle points in the phase plane (ϕ, θ), while $(\phi_2, 0)$ is a focus.

On the other hand, if $\mathrm{R}(0) \neq 0$, then the critical points of the system are the root of the following equation

$$\mathrm{R}(0) - \mathrm{G}(\phi) = 0, \tag{6}$$

and these roots (ϕ_1^*, ϕ_2^*, ϕ_3^*) must be real roots and $\phi_1^* < \phi_2^* < \phi_3^*$ or the kink-soliton will not exist. This means that the heteroclinic trajectories of (5) exist only for some values of the kink-soliton velocity that correspond to

$$-\int_{-\infty}^{+\infty} \mathrm{R}(-w\phi_z)\,\phi_z dz = 0. \tag{7}$$

From (5) and the Bogomolnyi method[44] one can easily get the following relationship between the parameters of the system and the shape of the kink-soliton:

$$-\int_{\phi_1^*}^{\phi_3^*} \mathrm{R}\left(-w\sqrt{2\mathrm{U}(\phi)}\right) d\phi = 0. \tag{8}$$

A stationary kink-soliton exist if eq. (8) has a real solution with respect to w. This is satisfied by $v = 0$, $v = v_1 > 0$, and $v = v_2 = -v_1$, but $v = 0$ is unstable.

However, the solution is a kink-soliton only if

$$\mathrm{U}(\phi) + \int_{\phi_1}^{\phi_3} \mathrm{R}\left(-w\sqrt{2\mathrm{U}(\phi)}\right) d\phi > 0, \tag{9}$$

for $\phi_1 < \phi < \phi_3$. In other words, if $\mathrm{R}(\phi_t)$ possesses two local extrema, such that $|\mathrm{R}(\phi_t)|$ at these extrema is R_M; if this value is comparable with the absolute value at the extrema of $\mathrm{G}(\phi)$ in the interval $\phi_1 < \phi < \phi_3$, then condition (9) is not satisfied.

2.2. Kink-soliton Stability

Now that we have given conditions for the strength of the external perturbation and nonlinear dissipation, we would like to study the stability of kink-soliton in the following system:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} - \mathrm{G}(\phi) = \mathrm{F}(x). \tag{10}$$

We are interested in the stability of static kink-soliton at the equilibrium position created by the inhomogeneous force $\mathrm{F}(x)$ [18]. For this, we will assume that the solution could be given as $\phi(x,t) = \phi_k(x) + f(x)\mathrm{e}^{\lambda t}$. Where $\phi_k(x)$ is a function with the topological and general properties of a kink, and a solution of

$$-\phi_{xx} - \mathrm{G}(\phi) = \mathrm{F}(x), \tag{11}$$

and $f(x)\mathrm{e}^{\lambda t}$ is a perturbation around $\phi_k(x)$. Therefore, from eq. (10) we can obtain the following spectral problem:

$$\widehat{L}f(x) = \Gamma f(x), \tag{12}$$

where $\widehat{L} = -\partial_{xx} - [\partial G(\phi)/\partial\phi]_{\phi=\phi_k}$, $\Gamma = -(\lambda^2 + \gamma\lambda)$; this allows us to determine the eigenvalues and eigenfunction of the discrete spectrum. This discrete spectrum corresponds to the kink-soliton modes (translational and internal *shapes* modes). We make use of the eigenvalues to find conditions for the kink-soliton stability.

3. Kink-soliton Internal Modes

In general, nonintegrable soliton equations (e.g. the φ^4 equation and the double sine-Gordon) may possess internal degrees of freedom, which are crucial in many phenomena [45–48]. A recent discussion of internal modes of solitary waves can be found in Ref. [49].

In some cases the internal modes can appear due to the discretization of the continuum equations [50–52].

However, it is well-known that the unperturbed ("*pure*"), continuum sine-Gordon equation does not have internal modes.

The sine-Gordon kink-solitons are very important in physics. They possess crucial applications in both particle physics and condensed matter theory. For instance, in solid state physics, they describe domain walls in ferromagnets, dislocations in crystals, charge density waves, fluxons in long Josephson junctions and Josephson transmission lines, etc. [1; 2; 5; 7–9]

Can external forces create internal modes in the sine-Gordon equation?

Recently there was a debate in the scientific literature about the existence of internal modes of sine-Gordon solitons. Some authors [53–59] have claimed that they have found an internal quasimode described as a long-lived oscillation of the width of the sine-Gordon soliton.

On the other hand, at the beginning of this century, an interesting paper was contradicting all these reports [60]. By considering the response of the soliton to ac forces and initial distortions, Quintero *et al* [60] showed that neither intrinsic internal modes nor "quasi-modes" exist in contrast to previous reports. We should stress that they use only time-dependent perturbations in their work.

In the present section we study the sine-Gordon equation perturbed by *spatiotemporal* external forces:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = \mathrm{F}(x,t). \tag{13}$$

We found that with some spatially inhomogeneous forces, the internal modes can exist for the sine-Gordon equation.

3.1. Spatially Inhomogeneous External Forces

We have shown (see Section 2.2.) that in equations as the following:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = \mathrm{F}_1(x), \tag{14}$$

if the force $\mathrm{F}_1(x)$ possesses a zero at x^* ($\mathrm{F}_1(x^*) = 0$), this can be an equilibrium position for the kink-soliton. If there is only one zero, this is a stable equilibrium position for the kink if $(\partial\mathrm{F}_1(x)/\partial x)_{x=x^*} > 0$. For the antikink, it is stable if $(\partial\mathrm{F}_1(x)/\partial x)_{x^*} < 0$.

Let us assume that $\mathrm{F}_1(x)$ is defined as:

$$\mathrm{F}_1(x) = 2(B^2 - 1)\sinh(Bx)/\cosh^2(Bx). \tag{15}$$

This is a function with a zero at $x^* = 0$.

We have chosen this function because it possesses the following properties:

- The exact solution for the soliton resting on the equilibrium position can be obtained.
- The stability problem of this soliton can be solved exactly.

The results obtained with this function can be generalized qualitatively to other systems topologically equivalent to this one. Besides, real physical systems are related to this example [5]. For instance, in a Josephson junction a perturbation that can be described by a function of type $\mathrm{F}(x) = d\mathrm{P}(x)/dx$, where $\mathrm{P}(x)$ is a bell-shaped function, is an Abrikosov vortex lying in the junction's plane perpendicular to its local dimension [61].

A similar function can describe a local deformation of a Charge Density Wave system [62].

Usually, function $\mathrm{F}(x)$ is taken as the Dirac's δ-function. However, if we wish to model a finite-width inhomogeneity, function (15) is a better choice.

The exact stationary solution of Eq. (14), with $\mathrm{F}_1(x)$ as defined in (15), is the following:

$$\phi_k = 4\arctan\left[\exp\left(Bx\right)\right]. \tag{16}$$

The stability analysis (see Section 2.2.), which considers small amplitude oscillations around ϕ_k $\left[\phi(k,x) = \phi_k(x) + f(x)e^{\lambda t}\right]$, leads to the eigenvalue problem [18; 27; 42; 63–66]:

$$\widehat{L}f = \Gamma f, \tag{17}$$

where $\widehat{L} = -\partial_x^2 + \left[1 - 2\cosh^{-2}(Bx)\right]$ and $\Gamma = -\lambda^2 - \gamma\lambda$.

This problem can be solved exactly [67]. The eigenvalues of the discrete spectrum are given by the formula

$$\Gamma_n = B^2(\Lambda + 2\Lambda n - n^2) - 1, \tag{18}$$

where $\Lambda(\Lambda + 1) = 2/B^2$.

The integer part of Λ (i.e. $[\Lambda]$) yields the number of eigenvalues in the discrete spectrum (i.e. $n \leq \Lambda$), which correspond to the kink-soliton modes: this includes the translational mode Γ_0, and the internal or shape modes Γ_n with $n > 0$ [18; 27; 42; 63–66].

All this analytical study produces the following results (note that parameter B will be our control parameter):

- For $B^2 > 1$, the translational mode is stable and there are no internal modes.
- If $1/3 < B^2 < 1$, then the translational mode is unstable. However, still there are no internal modes.
- When $1/6 < B^2 < 1/3$, apart from the translational mode, there is one internal mode! This internal mode is stable.
- In the case that $B^2 < 1/6$ there can appear many other internal modes! The exact number is $[\Lambda] - 1$, where $\Lambda(\Lambda + 1) = 2/B^2$.
- For $B^2 < 2/(\Lambda_*(\Lambda_*+1))$, where $\Lambda_* = (5+\sqrt{17})/2$, the first internal mode becomes unstable!

3.2. Cauchy Problem. Numerical Experiments

What happens when we shift the kink-soliton center-of-mass away from the equilibrium position?

We have the following initial problem:

$$\phi(x, 0) = 4 \arctan\{\exp[B(x - x_0)]\}, \tag{19}$$

$$\phi_t(x, 0) = 0. \tag{20}$$

In the stable case ($B^2 > 1$) the center-of-mass of the soliton will make damped oscillations (for $x_0 \neq 0$) around the equilibrium point $x^* = 0$ (see Fig. 2).

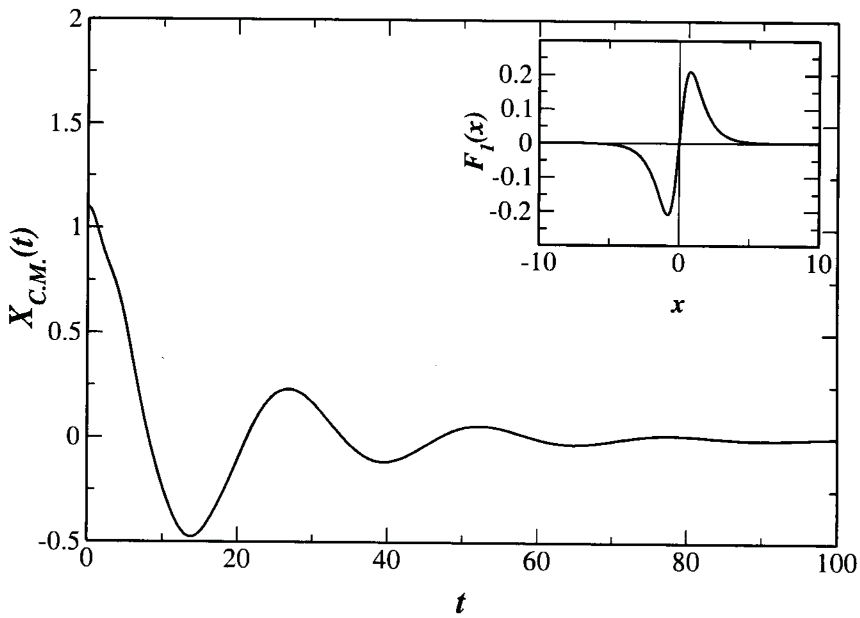

Figure 2. Soliton center of mass dynamics for the stable case ($B^2 > 1$). The inset shows $F_1(x)$ for $B^2 > 1$.

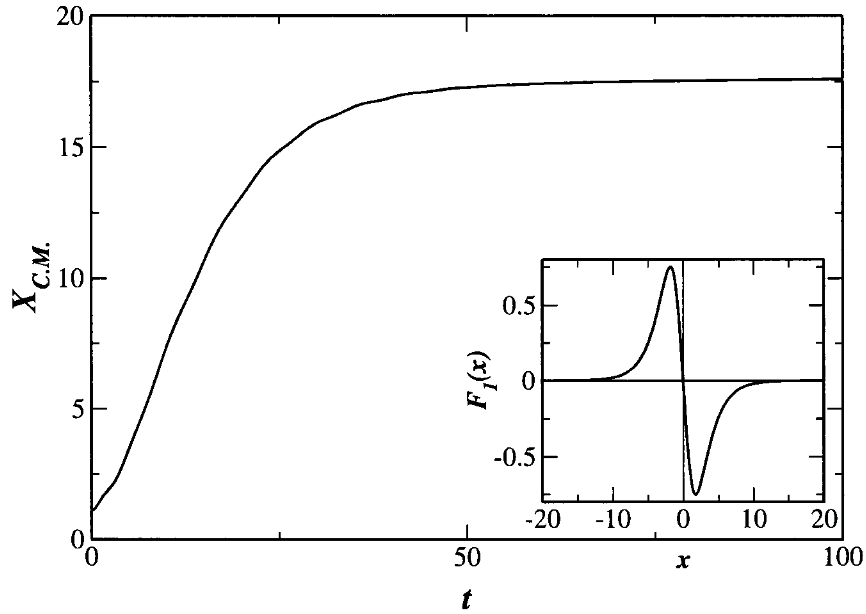

Figure 3. Soliton center of mass dynamics for the unstable case ($B^2 < 1$). The inset shows $F_1(x)$ for $B^2 < 1$.

In the case where the translational mode is unstable ($1/3 < B^2 < 1$), the soliton will move away indefinitely from the equilibrium position (see Fig. 3).

Consider the next initial problem:

$$\phi(x,0) = 4\arctan\left[\exp\left(Bx\right)\right] + C\sinh(Bx)\cosh^{-\Lambda}(Bx), \tag{21}$$

$$\phi_t(x,0) = 0. \tag{22}$$

In this initial problem the initial soliton is deformed.

For $1/6 < B^2 < 1/3$ we will observe oscillations of the soliton width (see Fig. 4). This is due to the fact that an internal mode ($n = 1$) has been excited. Eventually, due to unavoidable errors in the initial conditions or to the energy exchange between the internal mode and the translational mode, the soliton will move away from the equilibrium position (remember that the equilibrium position is unstable for the soliton center-of-mass).

It is important to note that the instability of the translational mode does not mean instability of the soliton structure.

We have to say that the frequency of the oscillations observed in the numerical simulations coincides with the one obtained theoretically using equation (18). The frequency of the width oscillations can be obtained using the equations $\omega_1 = \sqrt{\Gamma_1}$, where $\Gamma_1 = B^2(3\Lambda - 1) - 1$.

The most spectacular phenomenon occurs for $B^2 < 2/(\Lambda_*(\Lambda_* + 1))$, $\Lambda_* = (5 + \sqrt{17})/2$. In this case, the first internal mode is unstable. If we study the evolution of the soliton from the initial conditions (21)-(22), we will observe the destruction of the soliton (see Fig. 5). Two solitons move away (in different directions) to "infinity" (or to the boundaries of the system and an antisoliton is formed in the place of the original soliton

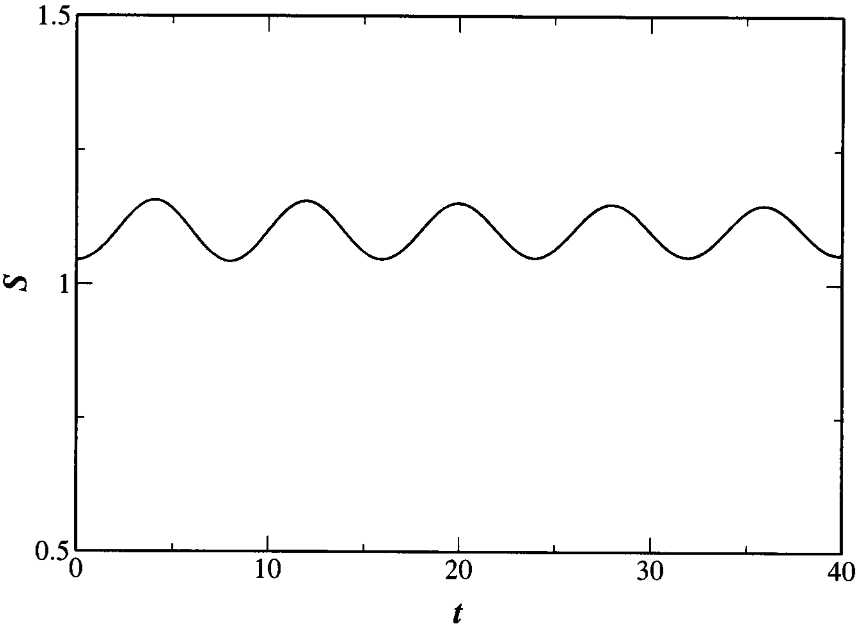

Figure 4. Soliton's width oscillations when the internal mode can be excited and it is stable, $\frac{1}{6} < B^2 < \frac{1}{3}$.

remaining there stabilized. In fact, the condition $B^2 < 1$ implies stability for the center-of-mass of an antisoliton.

Note that in this situations, the sine-Gordon solitons do not behave as rigid objects, which is expected from them in general [68].

3.3. Double-Well Potential for the Soliton

The initial distortions of the width of the soliton will eventually be damped due to dissipation.

Once the internal modes are possible, as in Eqs. (14)-(15) with $1/6 < B^2 < 1/3$, we need time-dependent external forces to sustain the oscillations of the soliton width.

On the other hand, if we wish the soliton to remain in some spatially localized zone, we need stable equilibrium positions for the center-of-mass of the soliton.

Let us consider the following spatiotemporal perturbation:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = \mathrm{F}_2(x) + \mathrm{F}_3(x,t), \tag{23}$$

where

$$\mathrm{F}_2(x) = \begin{cases} \mathrm{F}_1(x) & , \quad \text{if } -x_1 \leq x \leq x_1, \\ \dfrac{A}{\cosh\left[B\left(x + x_1\right)\right]} - D & , \quad \text{if } x < -x_1, \\ D - \dfrac{A}{\cosh\left[B\left(x - x_1\right)\right]} & , \quad \text{if } x > x_1, \end{cases} \tag{24}$$

and $\mathrm{F}_3(x,t) = f_0 \cos(\omega t) g(x)$, where $g(x) = 1/\cosh^2(E(x+x_1)) + 1/\cosh^2(E(x-x_1))$.

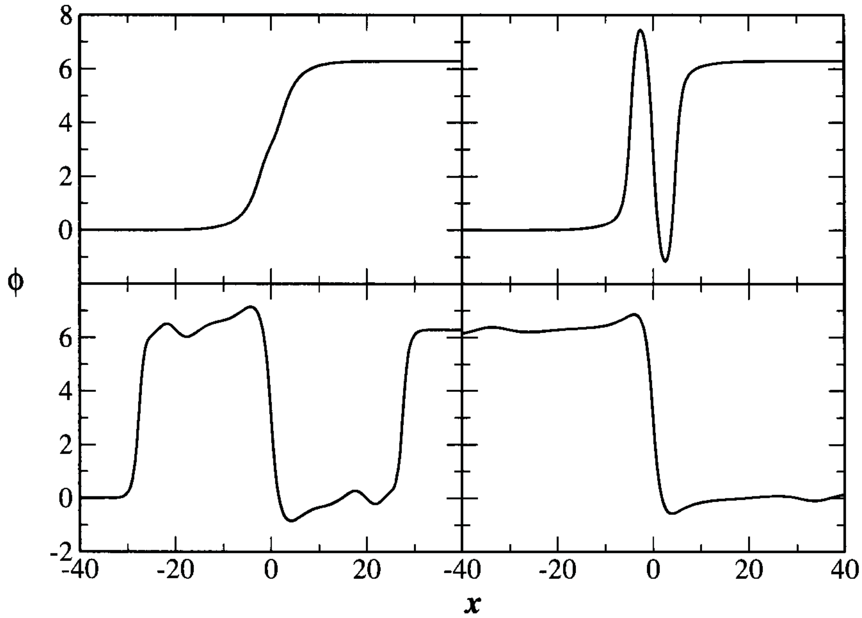

Figure 5. Soliton's destruction when the internal mode is unstable, $B^2 < \frac{2}{\Lambda_*(\Lambda_*+1)}$, where $\Lambda_* = \frac{5+\sqrt{17}}{2}$.

The space-dependent force $\mathrm{F}_2(x)$ creates a double-well potential for the soliton.

At the same time, in the interval $-x_1 < x < x_1$, we have the same force $\mathrm{F}_1(x)$, which was sufficient for the existence of the internal mode.

Actually, other forces can be used to excite the internal mode. In fact, if we have a function $\mathrm{F}(x)$ that can mimic approximately the behavior of function $\mathrm{F}_1(x)$ (specially in the interval $-x_1 < x < x_1$, where $-x_1$ and x_1 are the extrema of function $\mathrm{F}_1(x)$) when B satisfies the condition $1/6 < B^2 < 1/3$, then this function is good for exciting the internal mode. And note that the behavior of function $\mathrm{F}_1(x)$ in the interval $-x_1 < x < x_1$ is a very common behavior for a function in an interval where there is a zero and two extrema.

If in this double-well potential the middle (unstable) equilibrium position is such that the condition for the instability of the first internal mode holds, then we can observe again the destruction of the soliton (see Fig. 6).

As before, an antikink-soliton is created at position $x = 0$. However, in this case the "new" two solitons do not move away from the inhomogeneities. They remain trapped in the wells.

The time-dependent force $\mathrm{F}_3(x,t)$ will cause the soliton width to oscillate. The center-of-mass of the soliton will also oscillate.

In some cases, the soliton center of mass can oscillate, but these oscillations will remain confined inside one of the potential wells (see Fig. 7). On the other hand, if we slightly increase the amplitude of the driving force, we can make the soliton center of mass to jump between the potential wells created by force $\mathrm{F}_2(x)$ (see Fig. 8).

Although the soliton is not always in the interval $-x_1 < x < x_1$, it will return to this interval regularly. While the soliton is within this interval, all the conditions hold for the

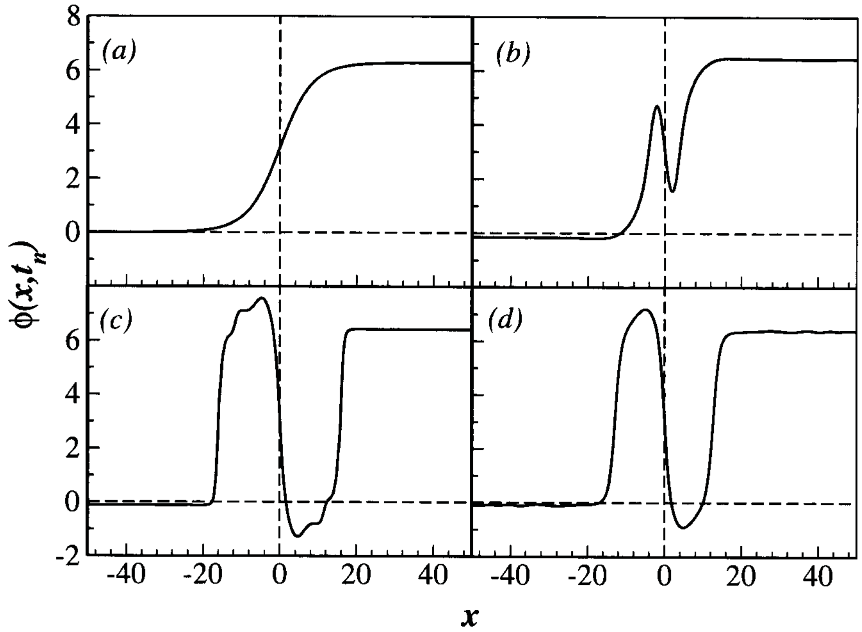

Figure 6. Soliton destruction inside the double-well potential (Eq. (23) with $B = 0.2,\ D = 0.0,\ \gamma = 0.1,\ C = -0.1,\ f_0 = 0.0,\ E = 0.2$).

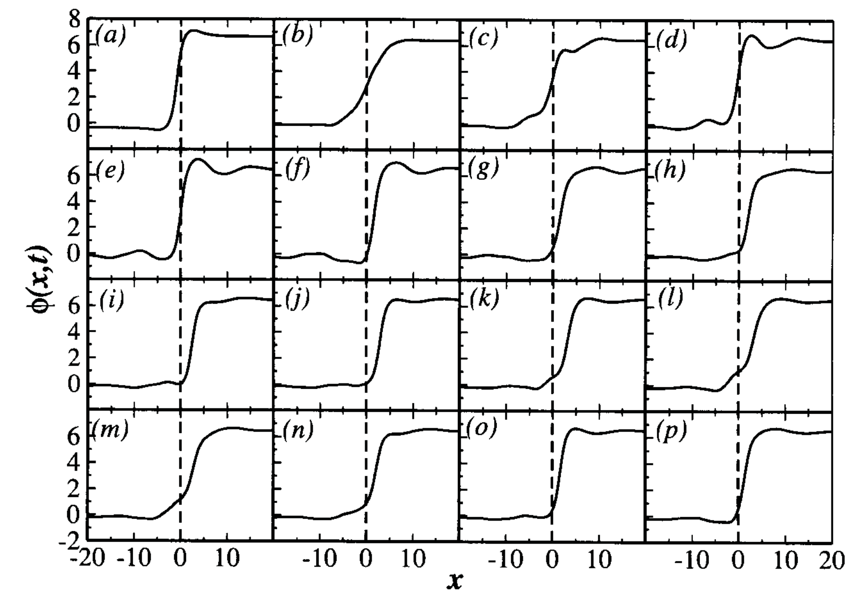

Figure 7. Soliton shape dynamics inside the right potential well (Eq. (23) with $B = 0.5,\ D = 0.2,\ x_0 = 0.0,\ v_0 = 0.0,\ \gamma = 0.1,\ C = 0.0,\ f_0 = 0.55,\ \omega = 0.55,\ E = 0.99$). The snapshots correspond to different time instants.

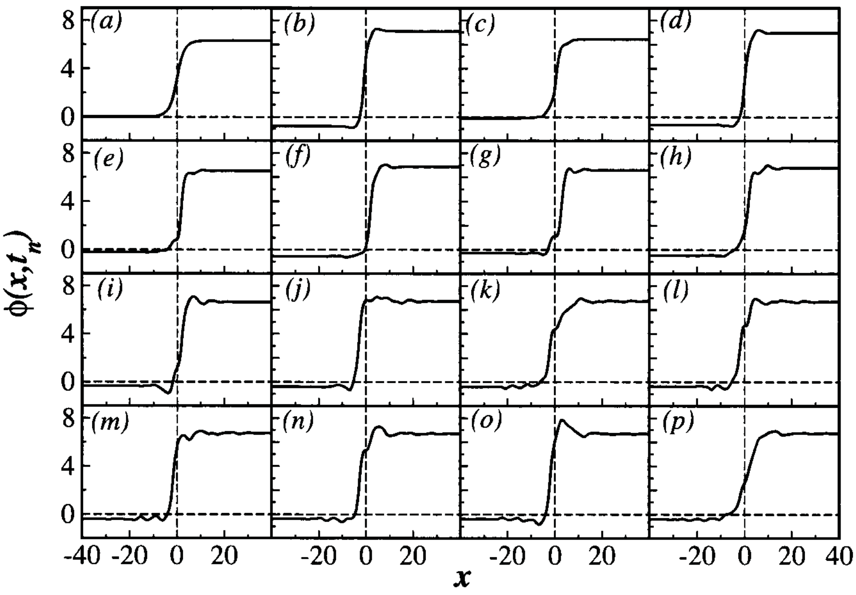

Figure 8. Soliton shape dynamics when the center of mass is jumping between the two wells (Eq. (23) with $B = 0.5,\ D = 0.4,\ x_0 = 0.0,\ v_0 = 0.0,\ \gamma = 0.1,\ C = 0.0,\ f_0 = 0.6,\ \omega = 1.0\ E = 0.8$).

internal mode to be excited.

A very small (further) increase in the amplitude of the driving force will cause extraordinary deformations of the soliton shape (see Fig. 9). Note that in some instances, the soliton can be destroyed, but in this case, this phenomenon is not permanent, because the soliton can be later reconstructed. This is a very dynamical structure with alternating destructions and reconstructions. The center of mass of the structure will be jumping between the potential wells.

In all these situations the initial soliton is not deformed. All the oscillations of the soliton width are produced by the time-dependent external forces.

We will now change the spatio-temporal force used to drive Eq. (23) with the following:

$$\mathrm{F}_3(x,t) = f_0 \cos(\omega t)\sinh(Bx)g(x) \tag{25}$$

where $g(x) = 1/\cosh^2(B(x+x_1)) + 1/\cosh^2(B(x-x_1))$. If f_0 is too large (see Fig. 10) the soliton will be immediately destroyed and the "three-particle" structure will be a permanent state. There will be oscillations in this structure but it always will be conformed by an antisoliton and two solitons.

When f_0 is not too large, we can have a situation similar to that of Fig. 9 (see Fig. 11) where there are large deformations of the soliton profiles with sporadic destructions of the soliton and eventual reconstructions.

The spatiotemporal force $\mathrm{F}_3(x,t)$ in Eq. (23) is chosen such that it acts directly on the internal mode. However, we have corroborated that other spatiotemporal forces can also sustain the soliton width oscillations. This includes the "ubiquitous" force $\mathrm{F}_3(t) = f_0 \cos(\omega\, t)$.

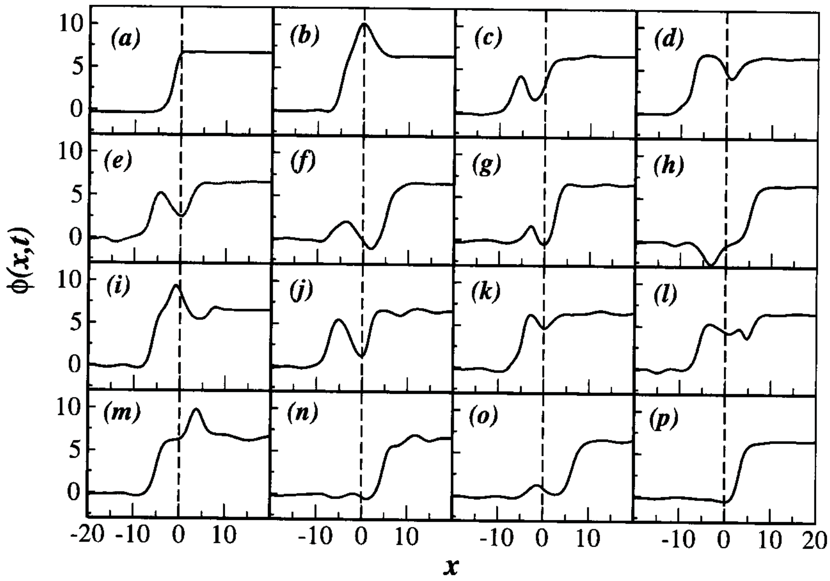

Figure 9. Soliton profiles for different time instants governed by Eq. (23) ($B = 0.5$, $D = 0.2$, $\gamma = 0.1$, $f_0 = 0.7$, $\omega = 0.55$, $E = 0.7$). Note the large deformations in soliton shape.

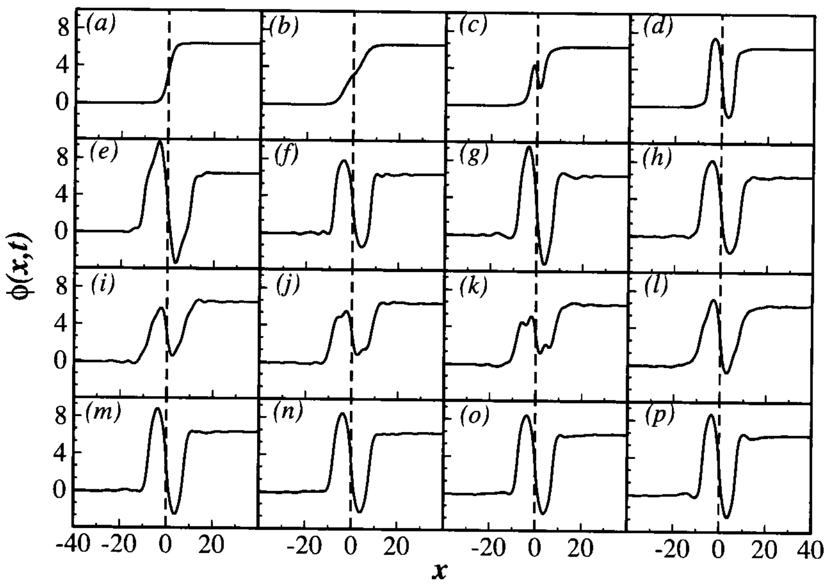

Figure 10. Permanent soliton destruction due to the action of the spatiotemporal force (25) which is associated to the first shape mode ($B = 0.5$, $D = 0.1$, $x_0 = 0.0$, $v_0 = 0.0$, $\gamma = 0.1$, $C = -0.1$, $f_0 = 0.4$, $\omega = 1.0$).

Here we should say that, by using the force (25) we can produce harmonic oscillations

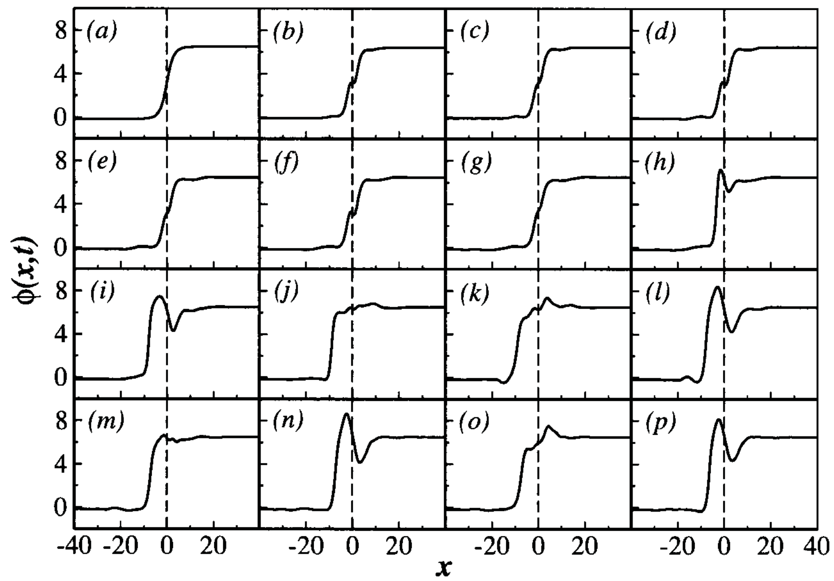

Figure 11. The same as in Fig. 10 but with a smaller amplitude of the driving force ($B = 0.5,\ D = 0.1,\ x_0 = 0.0,\ v_0 = 0.0,\ \gamma = 0.1,\ C = -0.1,\ f_0 = 0.2,\ \omega = 1.0$).

of the soliton width (see Fig. 12) even when the center of mass of the soliton is oscillating inside a stable potential well. This result was obtained using the following equation:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = \mathrm{F}_1(x) + \mathrm{F}_3(x,t), \tag{26}$$

where $\mathrm{F}_1(x) = B\mathrm{sinh}(Bx)/\mathrm{cosh}^2(Bx)$, $\mathrm{F}_3(x,t) = f_0\cos(\omega t)\mathrm{sinh}(Bx)/\mathrm{cosh}^2(Bx)$. Note that in general we will need in all cases spatiotemporal perturbations in order to produce oscillations of the shape modes. Particularly, in this case where the equilibrium position is stable, a spatiotemporal force that acts directly on the first internal mode is required.

Now we will return to the system where the soliton is moving in a double-well potential created by an external space-dependent force F_2 (24):

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = \mathrm{F}_2(x) + \mathrm{F}_3(x,t), \tag{27}$$

where $\mathrm{F}_3(x,t) = f_0\cos(\omega t)g(x)$, where $g(x) = 1/\mathrm{cosh}^2(E(x+x_1)) + 1/\mathrm{cosh}^2(E(x-x_1))$. In addition, $B = 0.5$, $D = 0.4$, $\gamma = 0.1$, $f_0 = 0.6$, $\omega = 1$, and $E = 0.8$. Here the parameters are chosen in such a way that the soliton center of mass will have a dynamics similar to a Duffing equation [69]:

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} - x + x^3 = f_0\cos(\omega t). \tag{28}$$

In this case, the soliton center of mass will perform chaotic oscillations jumping between the two wells created by force F_2 (see figures 13, 14). However, the most interesting result here (considering that our main subject is the internal mode) is the fact that we can observe chaotic oscillations of the soliton width (see Fig. 15).

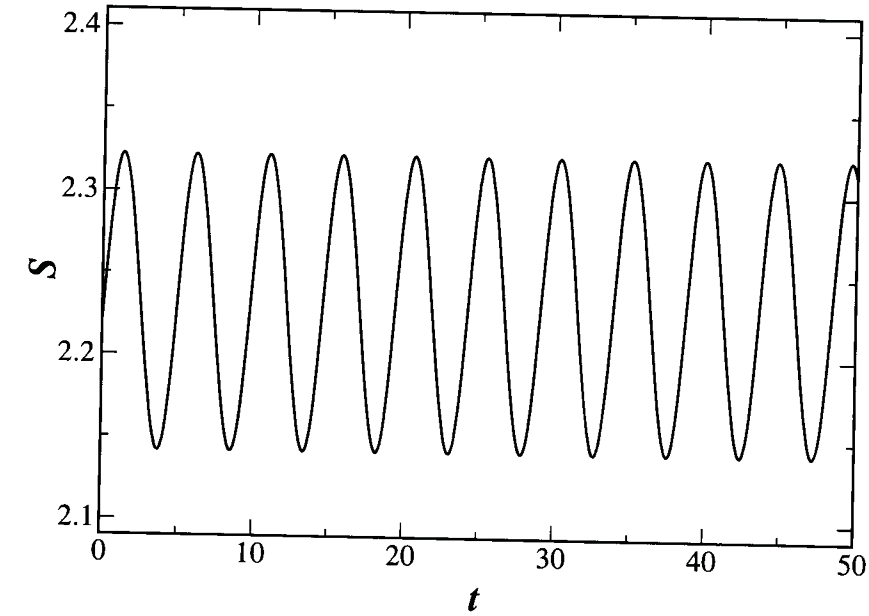

Figure 12. Time series of the soliton width for a one well potential, see Eq. (26) ($B = 0.5$, $\alpha = 0.0$, $f_0 = 0.1$, $\omega = 1.0$).

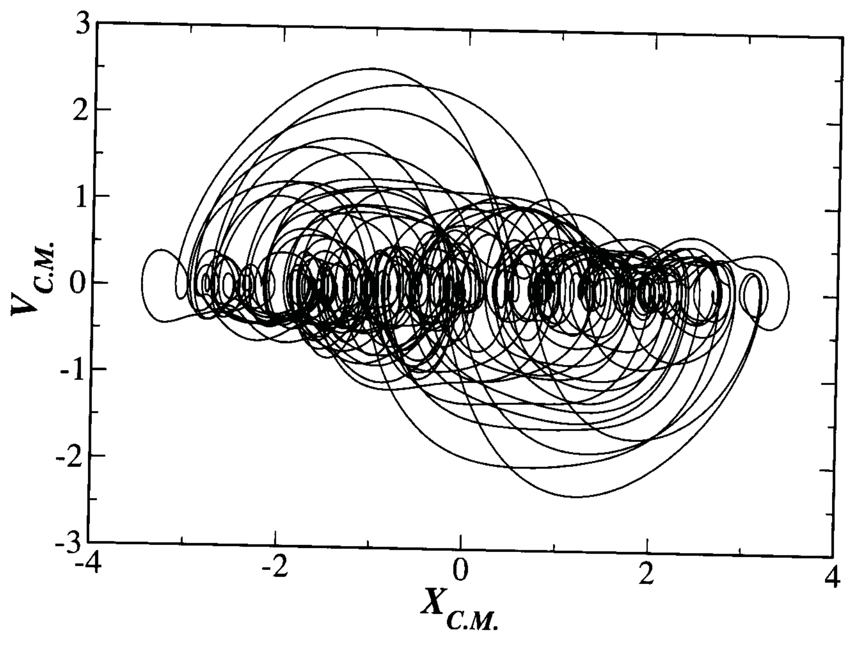

Figure 13. Phase portrait produced using the dynamics of the soliton center of mass governed by Eq. (27) ($B = 0.5$, $D = 0.4$, $x_0 = 0.0$, $v_0 = 0.0$, $\gamma = 0.1$, $C = 0.0$, $f_0 = 0.6$, $\omega = 1.0$, $E = 0.8$).

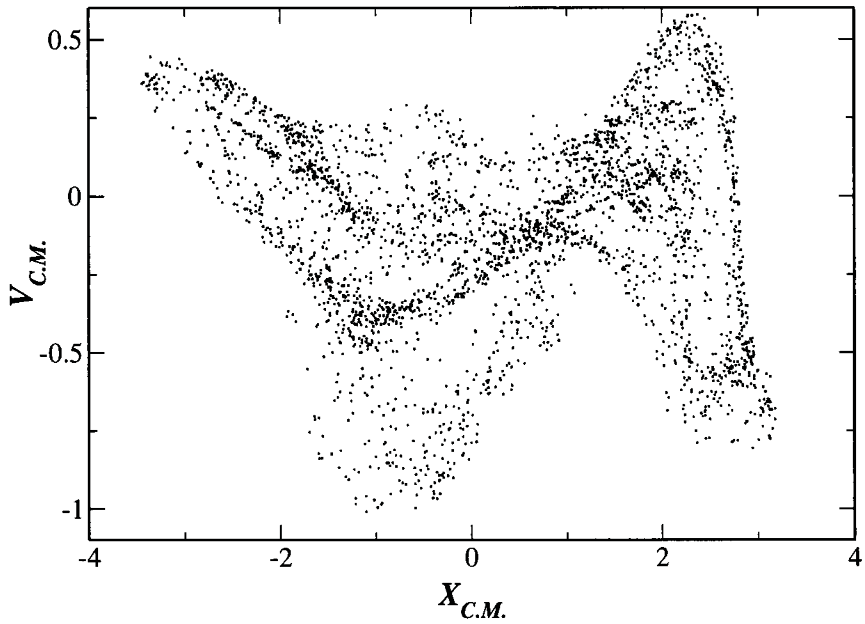

Figure 14. Poincaré map for the dynamics of Fig. 13 ($B = 0.5,\ D = 0.4,\ x_0 = 0.0,\ v_0 = 0.0,\ \gamma = 0.1,\ C = 0.0,\ f_0 = 0.6,\ \omega = 1.0\, E = 0.8$).

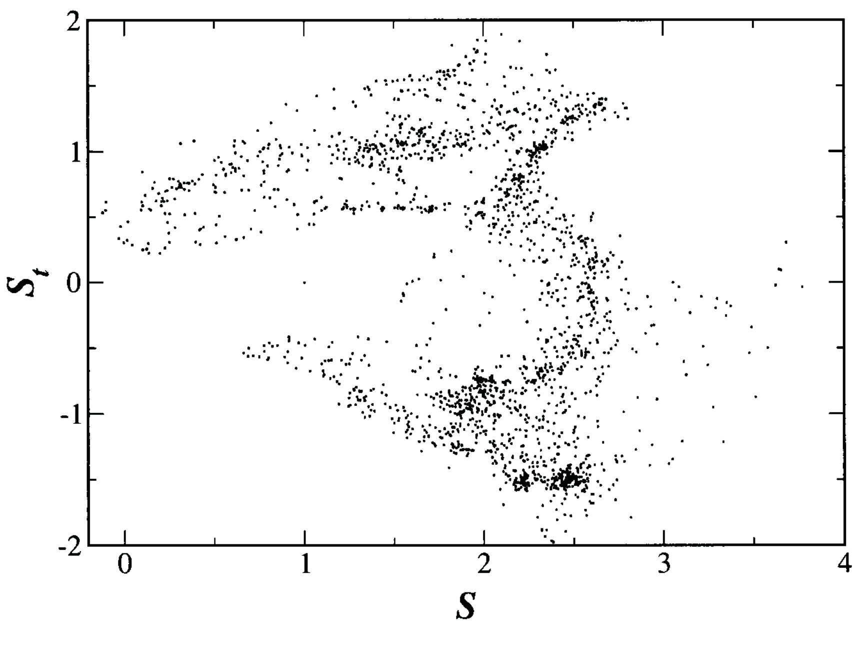

Figure 15. Poincaré map produced using the dynamics of the soliton width (see Eq. (27)) ($B = 0.5,\ D = 0.4,\ x_0 = 0.0,\ v_0 = 0.0,\ \gamma = 0.1,\ C = 0.0,\ f_0 = 0.6,\ \omega = 1.0$ $E = 0.8$).

3.4. Disordered Systems

The propagation of solitons in disordered media has been studied intensively in recent years [66; 70].

Consider the equation

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = \mathrm{F}(x). \tag{29}$$

where $\mathrm{F}(x)$ is defined in such a way that it possesses many zeroes, maxima and minima (see Fig. 16). This system describes an array of inhomogeneities.

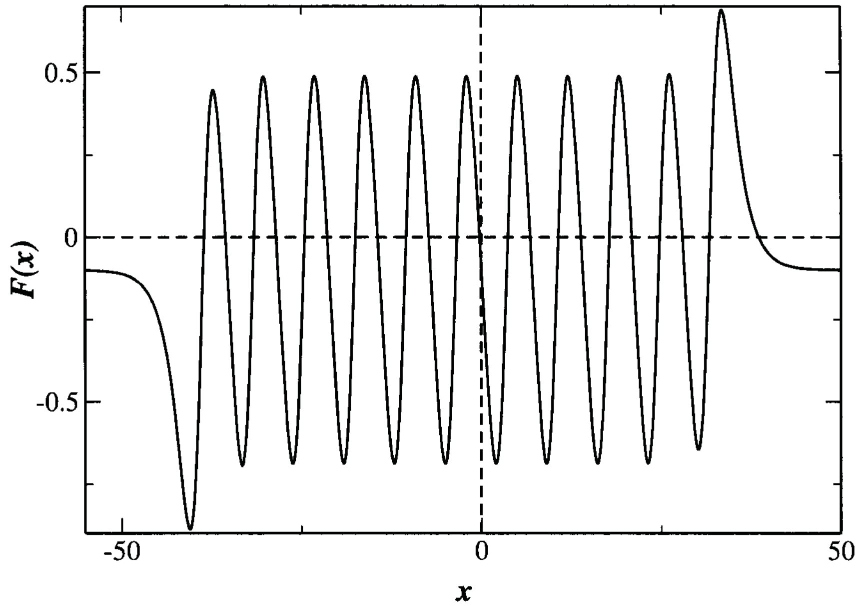

Figure 16. Array of inhomogeneities (see Eq. (30)).

For our study, we have defined $\mathrm{F}(x)$ in the following way:

$$\mathrm{F}(x) = \sum_{n=-q}^{q} 4\left(1 - B^2\right) \frac{\mathrm{e}^{B(x+x_n)} - \mathrm{e}^{3B(x+x_n)}}{\left(\mathrm{e}^{2B(x+x_n)} + 1\right)^2} \tag{30}$$

where $x_n = (n+2)\log\left(\sqrt{2}+1\right)/B$ ($n = -q, -q+1\cdots, q-1, q$), and $q+2$ is the number of extrema points of $F(x)$.

In our array, there is a "superposition" of the "disorder" with a dc component which will cause the soliton to move to the right all the time.

When the soliton is moving over intervals where $d\mathrm{F}(x)/dx < 0$, the internal mode can be excited. In fact, the points x_i where $\mathrm{F}(x_i) = 0$ and $d\mathrm{F}(x_i)/dx < 0$, are "barriers" which the soliton can overcome due to its kinetic energy. These "collisions" with the barriers will excite the internal modes if the function $\mathrm{F}(x)$ within these intervals mimics the behavior of $\mathrm{F}_1(x)$ when $1/6 < B^2 < 1/3$.

In Fig. 17 we can see how the width of the soliton will perform sustained oscillations during its motion in a disordered medium.

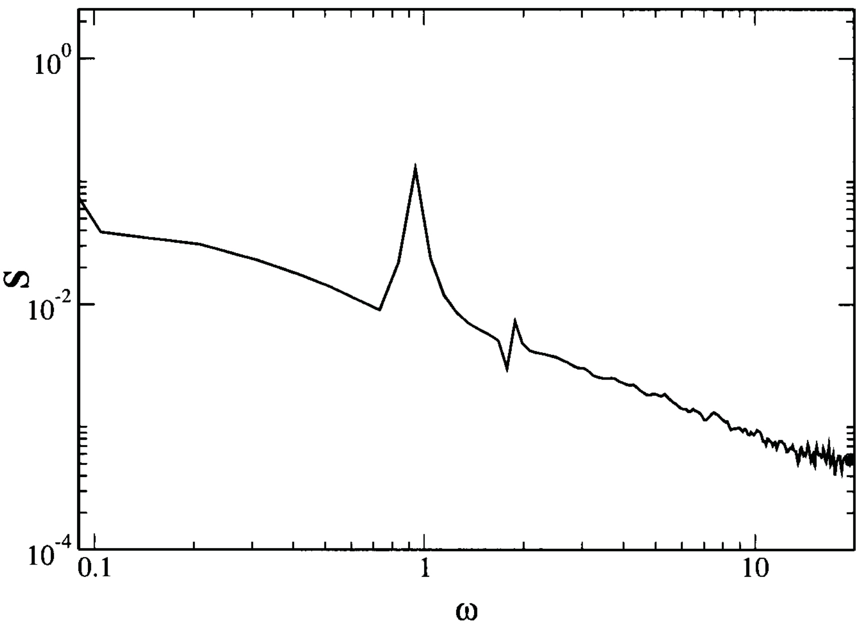

Figure 17. Fourier transform of the dynamics of the soliton width while it is moving in the array of inhomogeneities (see Eqs. (29) and (30))

4. Soliton Explosions

This section deals with the different mechanisms for kink-soliton explosions. We will show that in some cases, while some conditions hold, the kink-soliton explosion is permanent.

A kink-soliton destruction is observed when inhomogeneous space-dependent perturbations are present:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} - G(\phi) = F(x). \tag{31}$$

We should say that the zeroes of $F(x)$ are candidates for equilibrium positions for the soliton [18; 42; 43]. If $F(x)$ possesses only one zero x_0^* ($F(x_0^*) = 0$), then it is a stable position for the soliton if $(\partial F/\partial x)_{x_0^*} > 0$. Otherwise, the position is an unstable equilibrium. The opposite is true for the antisoliton. The center of mass of a soliton can make oscillations around a stable zero of $F(x)$, and it can move away from an unstable one. However, when $F(x)$ has an unstable zero at $x = x_0^*$ and additionally, the following conditions holds $\lim_{x\to\pm\infty} F(x) = \mp F_\infty$, $F_\infty > 0$ and F_∞ is larger than F_c, then the soliton can be destroyed and an antisoliton is "created" in the equilibrium position $x = x_0^*$.

When the soliton is in an unstable equilibrium position, it is "stretched" by the pair of forces that are acting on its body in opposite directions. And there is a limit for the magnitude of the pair of forces that the soliton can resist. Nevertheless, there is a more subtle mechanism for soliton destruction. When a soliton is close to an unstable equilibrium position, many internal shape modes of the soliton can be excited [18; 42; 43; 71]. If $(\partial^2 F/\partial x^2)_{x_0^*}$ is larger than some critical value, then the first internal shape mode can become unstable and this instability can lead to the soliton destruction. In this phenomenon the soliton decays into an antisoliton and two solitons. What is interesting in this situation is that $F(x)$ can be a localized perturbation.

Suppose we are interested in the stability of a soliton situated in equilibrium positions created by the inhomogeneous force $F(x)$. Using the method discussed in the previous section we construct an exact solution $\phi_k(x)$ with the topological properties of a kink-soliton. Then we investigate the stability of the solution solving the spectral problem (12).

Let us discuss some examples in detail.

Consider the following force

$$\mathrm{F}(x) = A\tanh(Bx)\left[D + E\,\mathrm{sech}^2(Bx)\right], \tag{32}$$

which can sustain a kink-soliton equilibrated at point $x = 0$.

When we solve the stability problem (12) we obtain the following eigenvalues of the discrete spectrum (for simplicity we assume $D = (A^2 - 1)/2$, $E = (4B^2 - A^2)/2$ and $G(\phi) = (\phi - \phi^3)/2$: $\Gamma_n = B^2(\Lambda + 2\Lambda n - n^2) - \frac{1}{2}$, where $\Lambda(\Lambda + 1) = 3A^2/2B^2$, $n < \Lambda$.

The translational mode of the soliton is stable if $2B^2\Lambda > 1$. If this condition is not satisfied, this just means that the kink-soliton center of mass will move away from the unstable equilibrium position $x = 0$. This does not necessarily lead to the soliton destruction.

However, if the following condition holds:

$$2B^2(3\Lambda - 1) < 1, \tag{33}$$

the kink-soliton first shape mode is unstable. In this case, the soliton can be destroyed!

In the very special (but also very illustrative) case $E = 0$, we have that for $4B^2 > 1$, the translational mode is stable. This means the equilibrium position $x = 0$ is stable for the soliton. The soliton center of mass can perform oscillations around point $x = 0$. If $4B^2 < 1$ (but $10B^2 > 1$), the translational mode is unstable. In this case, the soliton can move away from point $x = 0$, but it conserves its very characteristic shape because the internal modes are still stable. However, if $10B^2 < 1$, then the first internal (shape) mode becomes unstable. In this situation the soliton can explode. If we apply a spatiotemporal perturbation that periodically (in time) creates this instability in the place where the soliton is situated at that instant, then the result will be a highly nonstationary spatiotemporal state where the soliton is not allowed to recover its original shape. The soliton is in a permanent explosion.

Another very important example of equation (10) is the sine-Gordon equation (i.e. $\mathrm{G}(\phi) = -\sin\phi$). Suppose $\mathrm{F}(x) = 2(B^2 - 1)\sinh(Bx)\cosh^{-2}(Bx)$. This perturbation creates an equilibrium position for the sine-Gordon soliton at point $x = 0$. When we solve the eigenvalue problem (12) for the sine-Gordon soliton in the presence of this external force we get the following discrete spectrum: $\Gamma_n = B^2(\Lambda + 2\Lambda n - n^2) - 1$, where $\Lambda(\Lambda + 1) = 2/B^2$. The integer part of Λ yields the number of eigenvalues in the discrete spectrum. Additionally, the following conditions are found:

- For $B^2 > 1$, the translational mode is stable and there are no internal modes.
- If $1/3 < B^2 < 1$, then the translational mode is unstable (but still there are no internal modes).

- When $1/6 < B^2 < 1/3$ a stable internal shape mode appears.
- For $B^2 < 2/\left[\Lambda_* \left(\Lambda_* + 1\right)\right]$, where $\Lambda_* = \left(5 + \sqrt{17}\right)/2$, the first internal shape mode becomes unstable. This perturbation can destroy the sine-Gordon soliton.

We should say that a soliton, moving in a medium that is homogeneous everywhere except for a zone where the conditions for the instabilities hold, can undergo dramatic transient changes. But when the soliton leaves the mentioned zone, it will return to its original steady state shape.

How can we produce a permanent soliton explosion? Let us see what happen when time-dependent perturbations is used:

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} - \mathrm{G}\left(\phi\right) = \mathrm{F}_1\left(x\right) + \mathrm{F}_2\left(x, t\right), \tag{34}$$

where $\mathrm{F}_1\left(x\right)$ is a perturbation that creates a potential well for the soliton (i.e. $\mathrm{F}\left(x\right)$ possesses a zero x_0^* such that $\left(\partial\mathrm{F}/\partial x\right)_{x_0^*} > 0$) and $\mathrm{F}_2\left(x, t\right)$ is a space-time perturbation that periodically generates the instabilities conditions. Figure 18 shows an example of those highly complex spatiotemporal behaviors. In all the figures $\mathrm{G}\left(\phi\right) = \left(\phi - \phi^3\right)/2$. However, similar results are obtained with the sine-Gordon and other Generalized Klein-Gordon equations.

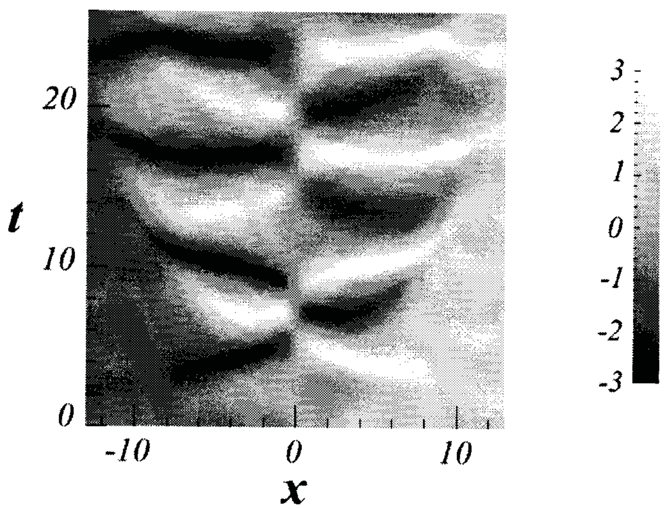

Figure 18. Initial steps of a permanent soliton explosion sustained by a spatiotemporal perturbation as in Eq. (34). Here $\mathrm{F}_1\left(x\right) = (1/2)A\left(A^2 - 1\right)\tanh\left(Bx\right)$, $\mathrm{F}_2\left(x, t\right) = (1/2)f_0 A\left(4B^2 - A^2\right)\cos\left(\omega t\right)\sinh\left(Bx\right)\cosh^{-3}\left(Bx\right)$, $A = 1.5$, $B = 0.2$, $\gamma = 0.1$, $\omega = 1$, $f_0 = 3$.

Can we produce permanent kink-soliton explosions without time-dependent external perturbations?

Let us discuss here briefly the importance of nonlinear damping. Linear dissipative systems like the damped harmonic oscillator $\phi_{tt} + \gamma\phi_t - \omega_0^2\phi = 0$ cannot sustain oscillations. However, the nonlinear oscillator $\phi_{tt} - b\phi_t + a\phi_t^3 + \omega_0^2\phi = 0$ supports a stable limit cycle

[69; 72]. The transition from a stable focus to an unstable focus and a stable limit cycle is the result of a Hopf bifurcation. This system is very easy to realize in practice using negative-resistance electronic elements [73].

Soliton systems as (31) can support solitons moving with a constant velocity. An example of this kind of systems can be realized in practice using a Josephson junction transmission line where the resistor is a negative-resistance twin-tunnel-diode circuit or a twin-transistor system [73]. In this case, $R(\phi_t) = -b\phi_t + a\phi_t^3$ is a good model!.

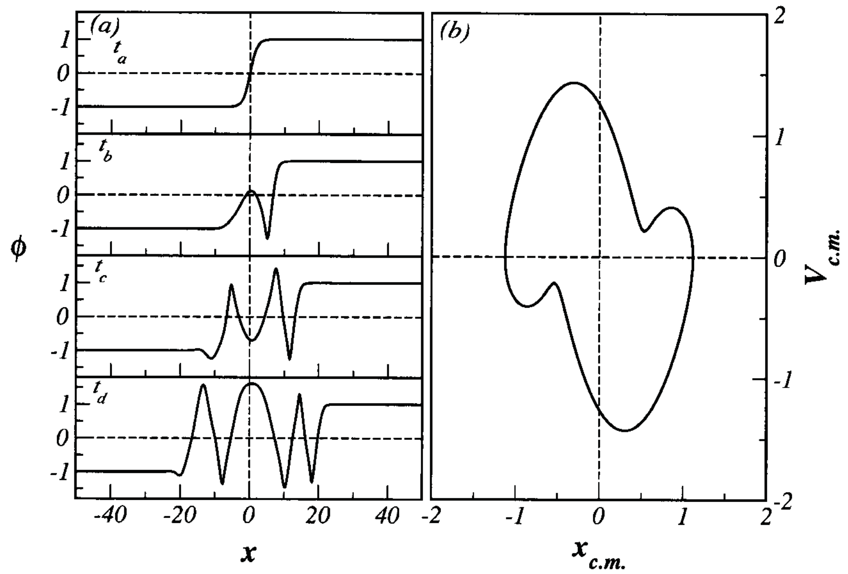

Figure 19. (a) Soliton explosion due to nonlinear damping. Here in Eq. (4): $R(\phi_t) = -b\phi_t + a\phi_t^3$, $a = 1$, $b = 0.7$. (b) Limit cycle produced with the dynamics of the soliton center of mass in Eq. (35). Here $\Gamma(x) = 1 - l/\cosh(Dx)$, $F(x) = A\tanh(Bx)$, $A = 0.45, B = 0.65, l = 2, D = 0.65$.

Let us investigated the shape mode stability of the soliton in the presence of nonlinear damping as we did before using the spectral problem (12).

We know from Section 2.1. that if R_m is comparable with the absolute value of the extrema of $G(\phi)$ (let us call it G_m), then the soliton can be destroyed. In fact, if $R_m > G_m$ the internal shape mode of the soliton can be unstable and the soliton becomes a highly nonstationary state.

When $F(x)$ in Eq. (10) has a stable zero, say $x = x_0^*$, the center of mass of a soliton can perform damped oscillations around x_0^*. If we wish to sustain these oscillations without explicit time-periodic external forces, then we should resort again to negative damping. Another way to experiment negative damping is when the damping coefficient in Eq. (4) is a function of x:

$$\phi_{tt} + \Gamma(x)\phi_t - \phi_{xx} - G(\phi) = F(x). \qquad (35)$$

Here $\Gamma(x)$ is negative in a neighborhood of x_0^* and positive elsewhere. This can be done in a chain of nonlinear oscillators using negative-resistance circuits [73] only in some small interval of the chain. An example of $\Gamma(x)$ with the required features is $\Gamma(x) = \gamma\left[1 - L/\cosh^2(Dx)\right]$, where $(1 - L) < 0$.

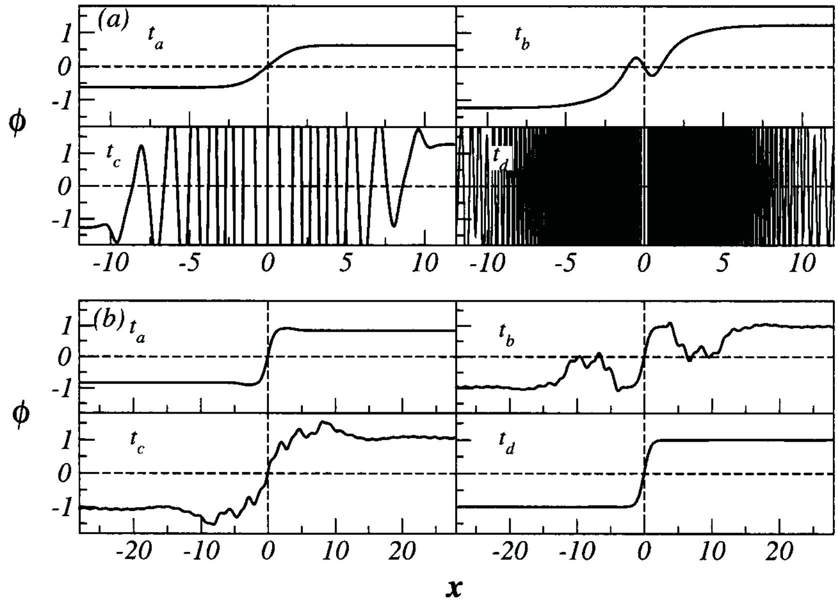

Figure 20. (a) Highly nonstationary spatiotemporal dynamics produced by Eq. (35). Here the simulated equation is the same as in Fig. 19 (b) with $l = 6$. (b) The soliton dynamics can be controlled in order to avoid the kink-soliton explosion (see Section 5.)

Figure 19 (b) shows a limit cycle which is the result of the dynamics of the kink-soliton center of mass in Eq. (35). However if we are not careful, the soliton can explode also in this system. We have solved the soliton stability problem for this equation. The most important result is that the first internal shape mode is unstable for $L > 5/2$. This behavior can be observed in Fig. 20 (a).

5. Pattern Control and Suppression of Spatiotemporal Chaos Using Geometrical Resonance

In Ref. [27] the concept of GR was used as a chaos-eliminating mechanism for the perturbed φ^4 equation. In this section we generalize the concept of Geometrical Resonance to a very general class of spatiotemporal systems which includes the sine-Gordon, Nonlinear Schrödinger, Boussinesq, Toda lattice and Complex Ginzburg-Landau equations (among others). We will use this concept as a method of chaos control when these equations are nonintegrable because of the presence of perturbations. GR is an extension of the linear notion of resonance to a nonlinear formulation based on a local energy conservation requirement [36].

5.1. Geometrical Resonance

Let us consider the partial differential equation

$$K_0[\phi] + K_1[\phi, x] = q(x,t)\, P[\phi], \tag{36}$$

where $K_0[\phi]$ and $P[\phi]$ are functions of ϕ, and its derivatives: ϕ_t, ϕ_x, ϕ_{tt}, ϕ_{xx}, etc. Equation $K_0[\phi] = 0$ is an integrable Hamiltonian system. This can be, for instance, the sine-Gordon equation (SGE), the Nonlinear Schrödinger equation (NLSE), the Boussinesq equation, or the Toda lattice [5; 7]. On the other hand, $K_1[\phi, x]$ includes dissipative terms and $q(x,t)\, P[\phi]$ is a very general driving force [18; 42; 43; 63].

At GR the amplitude, frequency, and space-time shape of $q(x,t)$ must satisfy some conditions so that some dynamical properties of the conservative system are preserved. We will call $\phi_{GR}(x,t)$ a GR solution of Eq. (36) if

$$K_1[\phi_{GR}, x] = q(x,t)\, P[\phi_{GR}]. \tag{37}$$

This implies a local energy conservation requirement. The energy integral that is conserved for equation $K_0[\phi] = 0$ is locally conserved for Eq. (36) if condition (37) holds. We can use this condition as a mechanism for chaos control when an additional condition holds: the GR solution must be an asymptotically stable solution of the (full) equation (36). This condition is introduced here for the first time. We will call Eq. (37) the exact GR condition and the solutions that satisfy this condition will be called GR solutions.

We can consider the energy of the system as a "local almost adiabatic invariant" [74]. Then we can write an approximate GR condition

$$\left\langle \frac{dH}{dt} \right\rangle_{T'} \simeq 0, \tag{38}$$

where H is the energy of the system and T' is the period of the chosen solution of equation $K_0[\phi] = 0$.

5.2. Sine-Gordon Equation

As an example, let us investigate the well-known driven and damped SGE

$$\phi_{tt} - \phi_{xx} + \gamma\phi_t + \sin\phi = q(x,t). \tag{39}$$

Suppose the task is to produce breathers of large amplitudes without entering a chaotic regime. The exact breather solution to the unperturbed SGE is

$$\phi(x,t) = 4\arctan\left[\frac{\sqrt{1-\omega^2}\sin(\omega t)}{\omega\cosh\left(\sqrt{1-\omega^2}x\right)}\right], \tag{40}$$

where ω is arbitrary in the interval $\omega^2 < 1$.

The external force $q(x,t)$ satisfies the GR condition (37) when

$$q(x,t) = q_{GR}(x,t) \equiv \frac{4\gamma\sqrt{1-\omega^2}\cos(\omega t)}{\cosh\left(\sqrt{1-\omega^2}x\right) + \left(\frac{1-\omega^2}{\omega^2}\right)\left[\frac{\sin^2(\omega t)}{\cosh\left(\sqrt{1-\omega^2}x\right)}\right]}. \tag{41}$$

In Eq. (39) if $q(x,t)$ is given by (41), the function (40) is an exact solution of the complete Eq. (39). When the parameters that define the perturbation (41) are fixed, there is only one frequency for which function (40) is the solution. This frequency is determined by that appearing in (41).

It is not difficult to show that the solution (40) is asymptotically stable in the framework of the full Eq. (39) with $q(x,t)$ given by Eq. (41). In the framework of the unperturbed SGE breathers form a continuum of solutions similar to the periodic solutions around the fixed points called centers in Dynamical Systems theory. These solutions are stable in the sense of Lyapunov but they are not asymptotically stable. However, the breather solution (40) in the framework of Eq. (39) with $q(x,t)$ given by Eq. (41) is a spatiotemporal limit cycle. That is, this is a spatiotemporal attractor. All close initial conditions (in all space configurations) for $t \to \infty$ will tend to behave as this solution. This phenomenon is shown in Fig. 21(a).

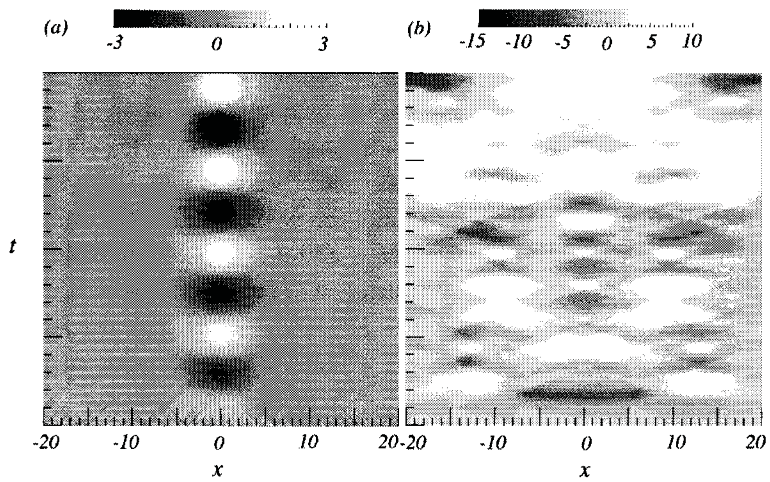

Figure 21. Robustness of the breather in Eq. (39). (a) Even with random initial conditions the breather is reorganized ($\omega = 0.707$, $\gamma = 0.45$). In the numerical simulations with the discretized equation the initial conditions were produced by a pseudorandom number generator of uniformly distributed values in the interval $[-1,1]$. (b) Irregular dynamics produced with a perturbation where the amplitude A, the frequency ω and the range Q are not close to satisfy the GR condition ($A = 4.5, \omega = 0.707, Q = 6.6$).

Perturbation $q_{GR}(x,t)$ can be approximated by a function of type $q(x,t) = f(t)g(x)$ where $g(x)$ is a bell-shaped function and $f(t)$ is a time-periodic function. This kind of perturbations has been used in several studies of the SGE [75; 76].

The general study of Eq. (39) using the GR concept and the breather solutions leads to the following conclusions: We can avoid chaos with amplitudes A of $q(x,t)$ for which $|A| \leq 4\gamma\sqrt{1-\omega^2}$ and $\omega^2 < 1$. On the other hand, the range Q of the function $g(x)$ (i.e. the interval of x where $g(x)$ is not exponentially small) should be $Q \leq \frac{1}{\sqrt{1-\omega^2}}$.

In some cases, when these conditions are not satisfied, the breather is not stabilized and we can get an irregular behavior. Figure 21(b) shows an example of the dynamics produced

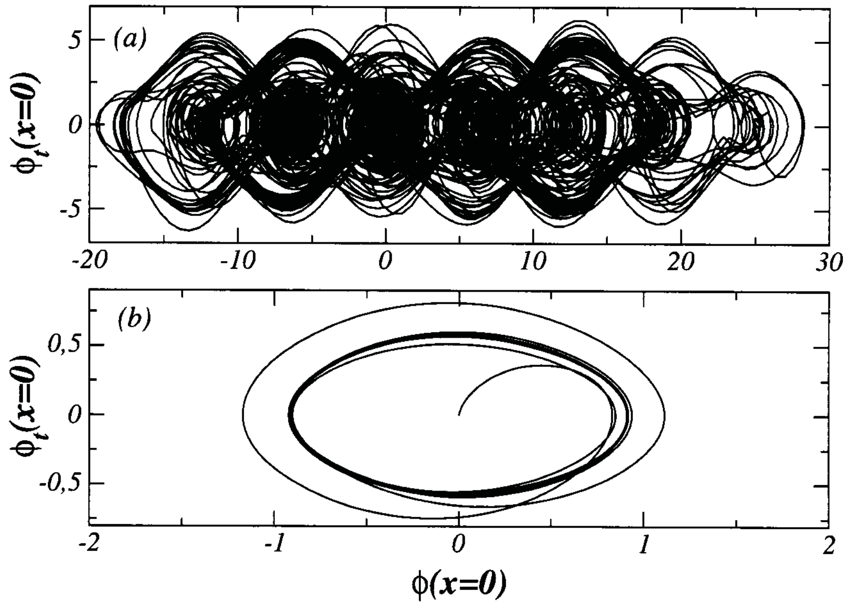

Figure 22. Suppression of spatiotemporal chaos using a given chaos-suppressing excitation in Eq. (42) ($f_0 = 0.91, \omega_d = 0.6$). (a) Well-developed spatiotemporal chaos when $f_c = 0$. (b) Controlled dynamics with $f_c = 0.4$, $\omega_c = 0.6$, $\theta = \pi/2$. Controlling spatiotemporal chaos using a localized excitation in Eq. (43): (c) Spatial profile for a given time moment corresponding to spatiotemporal chaos when $F_c(x,t) = 0$. (d)-(p) Spatial profiles for different time instants when $F_c(x,t)$ is given by Eq. (41), $\gamma = 0.1, f_0 = 0.5, \omega_d = 0.6, \omega = 0.6$. The time instants are (d) $t = 0.97$, (e) $t = 7.9$, (f) $t = 15.4$, (g) $t = 21$, (h) $t = 26.2$, (i) $t = 36.8$, (j) $t = 42$, (k) $t = 46.8$.

by Eq. (39) with

$$q(x,t) = A\cos(\omega t)\left[\cosh\frac{x}{Q} + \left(\frac{1-\omega^2}{\omega^2}\right)\left(\frac{\sin^2(\omega t)}{\cosh\frac{x}{Q}}\right)\right]^{-1}$$

where $A = 4.5, \omega = 0.707, Q = 6.6$.

There is a wealth of works [75; 76] dedicated to the numerical investigation of perturbed sine-Gordon equations using external forces of type $q(x,t) = f(t)g(x)$. All the results are in agreement with our theoretical results.

Sometimes we have the task of suppressing chaos using a given harmonic perturbation. Consider e.g. the following equation:

$$\phi_{tt} - \phi_{xx} + \gamma\phi_t + \sin\phi = f_0\sin(\omega_d t) + f_c\sin[\omega_c t + \theta], \tag{42}$$

where $f_0\sin(\omega_d t)$ is a chaos-producing excitation, while $f_c\sin[\omega_c t + \theta]$ is a chaos-suppressing excitation. The parameters of the chaos-suppressing excitation should be determined. In this case we can use the condition (38) to find parameters for the chaos-suppressing perturbation. Fig. 22 (a)-(b) shows an example of chaos control using this technique.

We would like to stress here that the force (41) can be used to control a well-developed spatiotemporal chaos.

Let us consider the following equation:

$$\phi_{tt} - \phi_{xx} + \gamma\phi_t + \sin\phi = f_0 \sin(\omega_d t) + F_c(x,t). \tag{43}$$

When $F_c(x,t) \equiv 0$, the system presents spatiotemporal chaos for $-\infty < x < \infty$, see Fig. 22(c). Now, if we turn on the controlling force $F_c(x,t)$ defined as function (41), we obtain a very regular spatiotemporal pattern as that shown in Fig. 22 (d)-(k). The most important remark here is that we are controlling spatiotemporal chaos in the whole space using a localized perturbation. We should add here that other works have used localized perturbations to control spatiotemporal chaos [77–79].

5.3. Nonlinear Schrödinger Equation

The damped and ac-driven NLSE

$$i\phi_t + \phi_{xx} + 2|\phi|^2\phi + i\alpha\phi = \varepsilon e^{i\omega t} \tag{44}$$

is another fundamental model in many areas of physics [80–82]. At sufficiently large ε the dynamics of this model becomes chaotic [82].

Suppose we have a general driving term: $i\phi_t + \phi_{xx} + 2|\phi|^2\phi + i\alpha\phi = q(x,t)$. We will take the one-soliton solution of unperturbed NLSE [7] as a GR solution: Then $q(x,t)$ must satisfy the condition:

$$q_{GR}(x,t) = \frac{\alpha\sqrt{\omega}e^{i(\omega t+\pi/2)}}{\cosh(\sqrt{\omega}x)}. \tag{45}$$

where ω can be any positive number. Thus, if the perturbation is localized and the amplitude ε satisfies the condition $\varepsilon \approx \alpha\sqrt{\omega}$, then the chaotic regime can be avoided. The one-soliton solution of NLSE is stabilized. We should remark that this can be achieved also by other localized perturbations.

If we use the two-soliton breather solution as a GR solution, we can obtain another driving force satisfying a GR condition:

$$q_{GR}(x,t) = \frac{4\alpha\left[\cosh(3x) + 3e^{i8t}\cosh(x)\right]e^{i(t+\pi/2)}}{\cosh(4x) + 4\cosh(2x) + 3\cos(8t)}. \tag{46}$$

See Ref. [7] for a discussion of multisoliton solutions. As in Eq. (45) we can introduce here an arbitrary parameter ω. However, in this case, we are more interested in the relationship between the two intrinsic frequencies of the solution. This force can be approximated by a function of type

$$F(x,t) = \varepsilon_1 g_1(x)e^{i(\omega_1 t+\pi/2)} + \varepsilon_2 g_2(x)e^{i(\omega_2 t+\pi/2)}, \tag{47}$$

where $\omega_1 = 1, \omega_2 = 9$, and $g_1(x)$ and $g_2(x)$ are localized functions.

In Ref. [82] a breather was stabilized using a two-frequency drive: $F(x,t) = \varepsilon_1 e^{i\omega_1 t} + \varepsilon_2 e^{i\omega_2 t}$, where $\omega_1 = 1$ and $\omega_2 = 9$. This result can be seen as a confirmation of the

GR approach for the NLSE. We should add that the phenomenon of breather stabilization is quite robust. For instance, if $\omega_1 = 1$, in addition to $\omega_2 = 9$, other close frequencies can be used, namely $\omega_2 = 8$ and $\omega_2 = 10$. This means that the GR condition can be satisfied approximately and that the Eq. (38) can also be used as a guide for the search of a controlling force. As in the case of the breather of the SGE, the breather of the NLSE is asymptotically stable.

Regarding localized excitations we should emphasize that GR analysis explains diverse fundamental results on stability of localized solutions previously obtained by perturbation theory [27; 42; 82] including those relative to one-soliton and two-soliton solutions of the NLSE. In this sense, in future works, it would be interesting to consider the case of the N-soliton solutions (see e.g. Appendix B, Ref. [7]).

5.4. Complex Ginzburg-Landau Equation

The control of spatiotemporal chaos (or turbulence) in the Complex Ginzburg-Landau equation (CGLE) [16; 25; 83–87] is a problem of great practical interest [87]. We are interested in the modified CGLE [25; 84]:

$$\phi_t = \phi + (1 + ic_1)\,\phi_{xx} - (1 - ic_3)\,|\phi|^2\,\phi + F_c\,(x,t)\,. \tag{48}$$

The term $F_c\,(x,t)$ is the control signal. Without the control signal $(F_c\,(x,t) = 0)$, the turbulence develops when the Benjamin-Feir condition $1 - c_1c_3 < 0$ is satisfied. This equation can be rewritten in the following form:

$$i\phi_t + c_1\phi_{xx} + c_3\,|\phi|^2\,\phi = i\left(\phi_{xx} + \phi - |\phi|^2\,\phi\right) + iF_c\,(x,t)\,. \tag{49}$$

When the right hand side of Eq. (49) is zero, it reduces to the NLSE.

If $\phi\,(x,t) = f\,(x)\exp\,(i\omega t)$ is a soliton solution of the NLSE, then we can use the following controlling signal

$$F_c\,(x,t) = \left[f^3\,(x) - f\,(x) - f_{xx}\,(x)\right]\exp\,(i\omega t)\,. \tag{50}$$

Equation (48) (with $F_c \equiv 0$) presents turbulence for $c_1 = 2$, $c_3 = 0.8$. We have been able to suppress this turbulence using the $F_c\,(x,t)$ given by Eq. (50) with $\omega = 12$ and $f\,(x)$ is the one-soliton solution of equation $c_1 f_{xx} - \omega f + c_3 f^3 = 0$ [7].

In this context, we should explain that in some cases, the stabilization process can require a force that is not a small perturbation. Furthermore, this technique can be used both as a way to stabilize a pre-existing solution of the unperturbed system and as a way to impose an arbitrary solution to the system. However, the success of all these endeavors depends on a very important fact: the final solution should be an asymptotical stable solution of the perturbed system. Incidentally, we should mention that the stabilization of unstable plane waves in the CGLE can be done using a nonlinear diffusion term [88].

6. Conclusion

We have shown that in sine-Gordon equations perturbed by inhomogeneous (space-dependent) forces $\mathrm{F}(x)$, the solitons can possess internal modes.

Our results are in agreement with previous works [60; 68]. In fact, in Eq. (14), with $F_1(x)$ as in (15), if we put $B^2 = 1$, then there are no external forces, and from Eq. (18) we obtain that there are no internal modes in that case neither.

Moreover, having any external inhomogeneous force does not necessarily mean having internal modes.

For instance, if $F(x)$ has a zero which corresponds to a stable equilibrium position for the soliton, even then the internal modes are impossible. This explains why it has been so difficult to find sine-Gordon internal modes.

For the existence of internal modes in the sine-Gordon solitons we need zeroes of the function $F(x)$ that corresponds to unstable positions for the center-of-mass of the soliton.

When the soliton center-of-mass is very close to an unstable equilibrium position, there is a pair of forces acting in opposite directions on the "body" of the soliton. This pair of forces should be sufficiently large to stretch the soliton "body", such that the soliton internal mode can be excited.

Function $F(x)$ can possess several zeroes corresponding to unstable and stable equilibrium positions. For instance, we have studied a force $F(x)$ that creates a double-well potential for the soliton.

Periodic time-dependent forces (along with the spatially inhomogeneous forces) can sustain the oscillations of the soliton width.

A soliton moving in an array of inhomogeneities can also undergo sustained oscillations of its width. All this is possible because the internal mode of the soliton can exist when it is moving in media where there are inhomogeneous space-dependent forces with unstable equilibrium positions.

Nonetheless, we have discovered another more remarkable phenomena. The sine-Gordon internal mode not only can exist for some external forces, but (in some situations) it can become unstable.

If we have an unstable equilibrium position for the soliton center-of-mass and the pair of forces acting on the soliton is too large, then the soliton can be destroyed. When the soliton is destroyed, it can be transformed into an antisoliton and two new solitons. The topological charge is conserved. We had found this phenomenon before for the φ^4-equation [63; 64]. However, here for the first time we have shown not only that the sine-Gordon soliton internal mode can exist, but also that it can become unstable and destroy the soliton. This is a spectacular manifestation of the fact that the sine-Gordon soliton can behave as a deformable (non-rigid) object.

We should say here that an effective potential for the soliton with stable and unstable equilibrium positions can be created also with the following perturbation

$$\phi_{tt} + \gamma\phi_t - \phi_{xx} + \sin\phi = H(x)\sin\phi, \tag{51}$$

where the extrema of function $H(x)$ can correspond approximately to the equilibrium positions for the soliton (i.e. not the zeroes as in the case of force $F(x)$). With some appropriate functions $H(x)$, the internal mode of the soliton can be excited too.

In Ref. [5], and the references cited therein, we can find many physical applications for such a system.

It has been recognized that the internal mode is a fundamental concept for the nonintegrable soliton models [49]. The internal modes of solitons govern the resonant soliton collisions and the soliton-impurity interactions [45–48].

The question about the conditions for creating the soliton's internal mode has interested physicists very strongly [49–52; 60]. Several authors [49–52] have shown how the internal mode can appear due to either a small deformation of nonlinearity or a weak discreteness. However, this is the first time that external perturbations are shown to be able to create an internal mode for the sine-Gordon soliton.

The existence of the internal mode is very relevant to the soliton dynamics because it can temporarily store part (or all) of the soliton's kinetic energy, which can later be restored again in the translational mode. This is important in the soliton-antisoliton collisions [45–47].

The coupling between the translational and the internal modes is the governing factor in many contexts, such as the interaction with inhomogeneities, with thermal noise, and with external drivings.

The soliton internal shape modes can contribute to the thermodynamic properties of the collective soliton-phonon gas [89; 90].

Considering all these facts, we believe that the proof of the existence of sine-Gordon internal modes in the presence of spatiotemporal perturbations is a fundamental result.

There are occasions when we apply some perturbations using technological means in order to satisfy the stability conditions. Nevertheless, we should say that, very often, nature itself applies the controlling perturbations. There are many natural regimes described by the mentioned equations in the presence of perturbations where the resulting dynamics is not chaotic. Our results can provide an explanation for these phenomena.

Numerous observations and experiments show that elastic waves from natural phenomena and human-made machines may alter water and oil production [91]. In some cases wave excitation may appreciably increase the mobility of these fluids. A new technology [91] based on these experiments is used to stimulate the reservoir as a whole. Here seismic frequency waves are applied at the earth's surface by arrays of vibrators. Many of the phenomena involved in this effect are described by the equations discussed in this paper, namely: NLSE, SGE, Boussinesq equation and other equations of type (4) (see Refs. [92]). For the optimization of the method, it is necessary to sustain spatiotemporal nonlinear oscillations of the reservoir with some frequency and shape. Based on ideas related to the results presented in this paper we have designed a new technology using a specific geometrical arrangement of the surface vibrators.

The nonlinear PDE possess an infinite number of different solutions. Among them one can choose a feasible one in order to implement GR. Even if only a given type of perturbation is allowed due to technical limitations, it is always possible to use the approximate condition (38) as in the case of task (42).

The concept that links all the situations where we have been able to suppress chaos is based on the mutual cancellation of nonintegrable terms as described by equations (37) and (38). In other words, we should add some temporal perturbation in such a way that (at least approximately) both the dissipative and the total driving terms mutually cancel. A remarkable situation (which is a particular case of the general theory but, at the same time, is present in all the studied systems) is that of breather-like oscillations. These patterns can be stabilized using some spatially localized time-periodic perturbations, where the amplitude, the spatial range and the frequency must satisfy some relationship. However, this phenomenon is robust. A fine-tuning is not necessary. There is always a whole valid interval of values for the amplitude, range and frequency that produces qualitatively the same result.

The most common perturbation in scientific research is $F(t) = f_0 \cos(\omega t)$. However, nature is very rich in dynamical behaviors. Our work shows that using very general spatiotemporal perturbations $F(x,t)$ we can make the difference between regular or chaotic behavior. Using certain spatiotemporal perturbations $F(x,t)$ we can stabilize a breather or we can produce a turbulent dynamics. We have been able to control different patterns in the sine-Gordon, Nonlinear Schrödinger, and Complex Ginzburg-Landau equations. Each of these systems possesses wide applications in many areas of Physics. Furthermore we believe that these ideas can be applied to other systems.

A. Bellorín would like to thank CDCH-UCV for support under project PI-03.00.6388.2006. L. E. Guerrero would like to thank FONACIT for support under project G-2005000449.

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