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From exactly solvable chaotic maps to stochastic dynamics

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Abstract

For a class of nonlinear chaotic maps, the exact solution can be written as $X_n = P(\theta k^n)$, where $P(t)$ is a periodic function, θ is a real parameter and k is an integer number. A generalization of these functions: $X_n = P(\theta z^n)$, where z is a real parameter, can be proved to produce truly random sequences. Using different functions $P(t)$ we can obtain different distributions for the random sequences. Similar results can be obtained with functions of type $X_n = h[f(n)]$, where $f(n)$ is a chaotic function and $h(t)$ is a noninvertible function. We show that a dynamical system consisting of a chaotic map coupled to a map with a noninvertible nonlinearity can generate random dynamics. We present physical systems with this kind of behavior. We report the results of real experiments with nonlinear circuits and Josephson junctions. We show that these dynamical systems can produce a type of complexity that cannot be observed in common chaotic systems. We discuss applications of these phenomena in dynamics-based computation.

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1. Introduction

Chaos is now accepted to comprise a class of phenomena intermediate between regular sinusoidal (or quasiperiodic) and unpredictable, truly stochastic behavior [1–11]. Scientists have found applications of chaos theory in many scientific and practical situations [1–11].

The most studied chaotic systems (logistic map, Lorenz model, Duffing equation, etc.) are not random. In all these systems the previous values of a time-series determine the future values. What makes the long-time behavior of chaotic systems difficult to predict is the sensitive dependence on initial conditions.

In the truly stochastic systems, past sequences of values do not give enough information to determine even the next value. Recently, several methods have been devised for distinguishing chaos from random time series. Some of these methods are based on nonlinear forecasting [12].

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Recently, we have introduced explicit functions that produce truly random sequences [13–15]. For instance, let us define the function:

$$X_n = \sin^2(\theta \pi z^n), \tag{1}$$

where z is a real number and θ is a parameter.

For an integer $z > 1$, this is the solution to some chaotic maps [13–15]. For a noninteger z, function (1) can produce truly unpredictable random sequences whose values are independent.

Function (1) with noninteger z cannot be expressed as a map of type

$$X_{n+1} = f(X_n, X_{n-1}, \ldots, X_{n-r+1}). \tag{2}$$

In the present paper we address the following questions: What are the properties of generalized functions of type $X_n = P(\theta z^n)$ which lead to random dynamics? Can an autonomous dynamical system with several variables produce a random dynamics similar to that of function (1)? We will present several dynamical systems with this kind of behavior. We will report the results of real experiments with nonlinear circuits, which contain direct evidence for this new phenomenon. We discuss some applications.

We will study very general functions of type $X_n = P(\theta z^n)$, where z is a real number. We will show that (provided some conditions are met) these functions can produce random sequences. Using different functions $P(t)$ we can obtain different distributions for the random sequences. We will also investigate functions of type $X_n = h[f(n)]$, where $f(n)$ is a finite nonperiodic oscillating function which possesses repeating intervals with finite exponential behavior, and $h(t)$ is a noninvertible function. We will show that a dynamical system consisting of a chaotic map coupled to a map with a noninvertible nonlinearity can generate random dynamics. We will present several dynamical systems with this kind of behavior. We will report the results of real experiments with nonlinear circuits and Josephson junctions. We will show that these dynamical systems can produce a type of complexity that cannot be observed in common chaotic systems. We discuss applications of these phenomena in dynamics-based computation.

2. Random sequences

It is important to say that the function $X_n = \sin^2(\theta \pi 2^n)$ is the general solution to the logistic map $X_{n+1} = 4X_n(1 - X_n)$ [16–18]. Recently, many outstanding papers have been dedicated to exactly solvable chaotic maps [19–25]. In some of these papers the authors not only find the explicit functions X_n that solve the maps, but also they discuss the statistical properties of the sequences generated by the chaotic maps.

For many of the one-dimensional maps discussed in the mentioned papers the exact solution can be written as

$$X_n = P(\theta k^n), \tag{3}$$

where $P(t)$ is a periodic function, and k is an integer.

In the present paper, we will consider a sequence $X_0, X_1, X_2, \ldots,$ unpredictable if given any string of $m + 1$ values $X_0, X_1, X_2, \ldots, X_m$, the next value X_{m+1} is not determined by the previous values.

Under this definition, any one-dimensional map of type $X_{n+1} = f(X_n)$ cannot produce an unpredictable dynamics. In particular, the logistic map is not unpredictable: even an r-dimensional map of type $X_{n+1} = f(X_n, X_{n-1}, \ldots, X_{n-r+1})$ is not unpredictable.

In the present paper we will study functions of type:

$$X_n = P(\theta z^n), \tag{4}$$

where z is a real number.

We will show that the dynamics produced by these functions (for generic z) is very different from the dynamics produced by functions of type (3) where k is an integer. In fact, there is no map of type $X_{n+1} = f(X_n, X_{n-1}, \ldots, X_{n-r+1})$ able to produce such a dynamics.

Let us now discuss some properties of function (1). We will present here a short proof of the fact that the sequences generated by function (1) are unpredictable from the previous values. This proof is presented here for the first time. However, a more detailed discussion of the properties of these functions (including statistical tests) can be found in [13–15]. Let z be a rational number expressed as $z = p/q$, where p and q are relative prime numbers.

We are going to show that if we have $m + 1$ numbers generated by function (1): $X_0, X_1, X_2, X_3, \ldots, X_m$ (m can be as large as we wish), then the next value X_{m+1} is still unpredictable. This is valid for any string of $m + 1$ numbers.

Let us define the following family of sequences:

$$X_n^{(k,m)} = \sin^2\left[\pi(\theta_0 + q^m k)\left(\frac{p}{q}\right)^n\right], \tag{5}$$

where k is an integer. The parameter k distinguishes the different sequences.

For all sequences parameterized by k, the first $m + 1$ values are the same. This is so because

$$X_n^{(k,m)} = \sin^2\left[\pi\theta_0\left(\frac{p}{q}\right)^n + \pi k p^n q^{(m-n)}\right] = \sin^2\left[\pi\theta_0\left(\frac{p}{q}\right)^n\right] \tag{6}$$

for all $n \le m$. Note that the number $kp^n q^{(m-n)}$ is an integer for $n \le m$. So we can have infinite sequences with the same first $m + 1$ values.

Nevertheless, the next value

$$X_{m+1}^{(k,m)} = \sin^2\left[\pi\theta_0\left(\frac{p}{q}\right)^{m+1} + \frac{\pi k p^{m+1}}{q}\right] \tag{7}$$

is uncertain.

In general, $X_{m+1}^{(k,m)}$ can take q different values. These q values can be as different as 0, 1/2, $\sqrt{2}/2$, $1/e$, $1/\pi$, or 1. From the observation of the previous values $X_0, X_1, X_2, X_3, \ldots, X_m$, there is no method for determining the next value.

This result shows that for a given set of initial conditions, there exists always an infinite number of values of θ that satisfy those initial conditions. The time-series produced for different values of θ satisfying the initial conditions is different in most of the cases. Even if the initial conditions are exactly the same, the following values are completely different. This property is, in part, related to the fact that the equation $\sin^2\theta = \alpha$, where $0 \le \alpha \le 1$, possesses infinite solutions for θ.

We should stress that from the observation of a string of values $X_0, X_1, X_2, X_3, \ldots, X_m$ generated by function (1) it is impossible to determine which value of θ was used.

For z irrational (we exclude the numbers of type $z = m^{1/k}$), the numbers generated by function (1) are completely independent. After any string of $m + 1$ numbers $X_0, X_1, X_2, X_3, \ldots, X_m$, the next outcome X_{m+1} can take infinite different values.

Of course, irrational numbers cannot be implemented in digital computers. However, in order to produce unpredictable dynamics, we do not need to use irrational numbers. As we have shown in this section, if $z = p/q$ in function (1), the produced dynamics is completely unpredictable. For any string of values $X_0, X_1, X_2, \ldots, X_m$, the next value X_{m+1} can take q different values. It is impossible to know which of these q values is the next. Thus, in order to produce unpredictability with function (1) it is enough to use rational numbers of type $z = p/q$.

If we wish to increase the uncertainty, then we should use a number $z = p/q$ (p and q relative primes) such that q is very large.

In any case, there is no map of type $X_{n+1} = f(X_n)$ or even of type $X_{n+1} = f(X_n, X_{n-1}, \ldots, X_{n-r+1})$ that could produce unpredictable dynamics as that generated by function (1) even with a rational z of type $z = p/q$. For all these maps, the next values are fully determined by the previous values.

For any given string of sequence values $X_0, X_1, X_2, \ldots, X_m$, we can always find infinite values of θ such that the corresponding sequences can possess this same string of $m + 1$ values, but the next values can be different.

The numbers produced by function (1) are random but are not distributed uniformly. The probability density behaves as $P(X) \sim 1/\sqrt{X(1-X)}$. If we need uniformly distributed random numbers, we should make the following transformation $Y_n = (2/\pi)\arcsin\sqrt{X_n}$. In this case $P(Y) = \text{const}$.

3. Generalized random functions

It is important to mention here that the argument of function (1) does not need to be exponential all the time, for $n \to \infty$. In fact, a set of finite sequences (where each element-sequence is unpredictable, and the law for producing a new element-sequence cannot be obtained from the observations) can form an infinite unpredictable sequence. See the discussion in the following paragraph.

So if we wish to produce random sequences of very long length, we can determine a new value of parameter θ after a finite number N of values of X_n. This procedure can be repeated for the desired number of times. It is important to have a nonperiodic method for generating the new value of θ. For example, we can use the following method in order to change the parameter θ after each set of N sequence values. Let us define $\theta_s = AW_s$, where W_s is produced by a chaotic map of the form $W_{s+1} = f(W_s)$ where s is the order number of θ in a way that $s = 1$ corresponds to the θ used for the first set of N values of X_n, $s = 2$ for the second set, etc. The inequality $A > 1$ should hold to ensure the absolute unpredictability. In this case, from the observation of the values X_n, it is impossible to determine the real value of θ.

After a carefully analysis of function (1), we arrive at the preliminary conclusion that (to produce unpredictable dynamics) the main characteristics for any functions are the following. The function should be able to be re-written in the form:

$$X_n = h(f(n)), \tag{8}$$

where the argument function $f(n)$ grows exponentially and the function $h(y)$ should be finite and periodic. This result allows us to generalize this behavior to other functions as the following:

$$X_n = P(\theta z^n), \tag{9}$$

where $P(t)$ is a periodic function. So there is nothing special in function $X_n = \sin^2(\theta\pi z^n)$. Let us see some examples of functions of type $X_n = P(\theta z^n)$.

The first-return maps produced by function

$$X_n = \sin(2\theta\pi z^n) \tag{10}$$

is shown in Fig. 1.

Note that in this case (if $z = p/q$) for a given X_n, we have $2q$ possible values for X_{n+1}; and for a given X_{n+1}, we have $2p$ possible values of X_n.

A different first-return map (see Fig. 2) can be generated by function

$$X_n = \cos(2\theta\pi z^n). \tag{11}$$

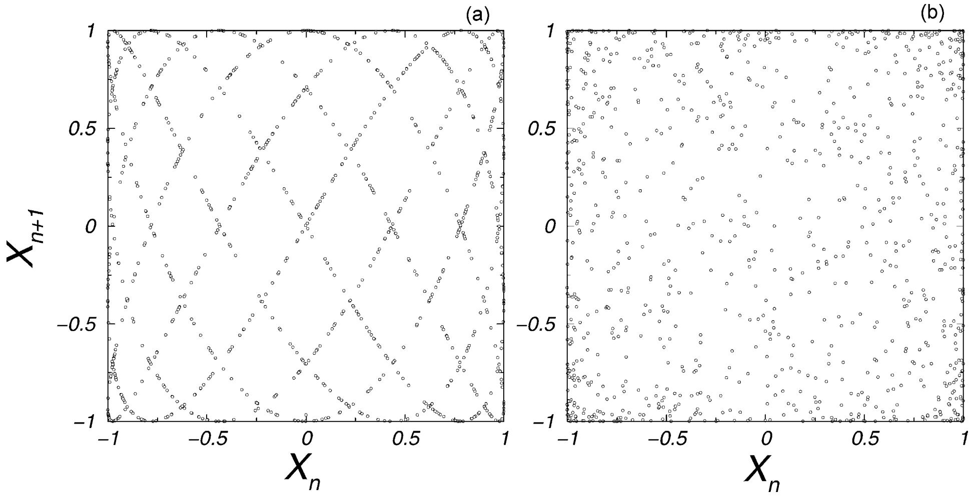

Fig. 1. First-return maps produced by function (10): (a) $z = 7/4$ and (b) $z = e$.

The distribution functions of Eqs. (10) and (11) are peaked at points $X_n = 0$ and 1. A generalization of the trigonometric functions (elliptic functions)

$$X_n = cn(\theta z^n, k) \tag{12}$$

is also good for the construction of stochastic functions (see Fig. 3).

These functions can be used for many algebraic manipulations that can lead to the solution of different stochastic problems.

A very special stochastic function is the following:

$$X_n = \theta z^n \pmod 1. \tag{13}$$

In Fig. 4 we can find the first-return maps. Function (13) can produce uniformly distributed random numbers.

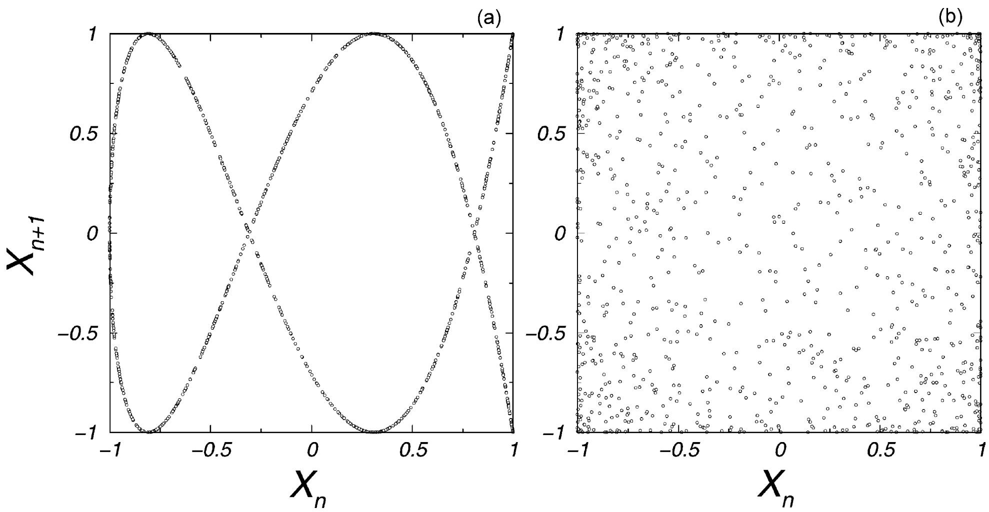

Fig. 2. First-return maps produced by function (11): (a) $z = 5/2$ and (b) $z = \pi/2$.

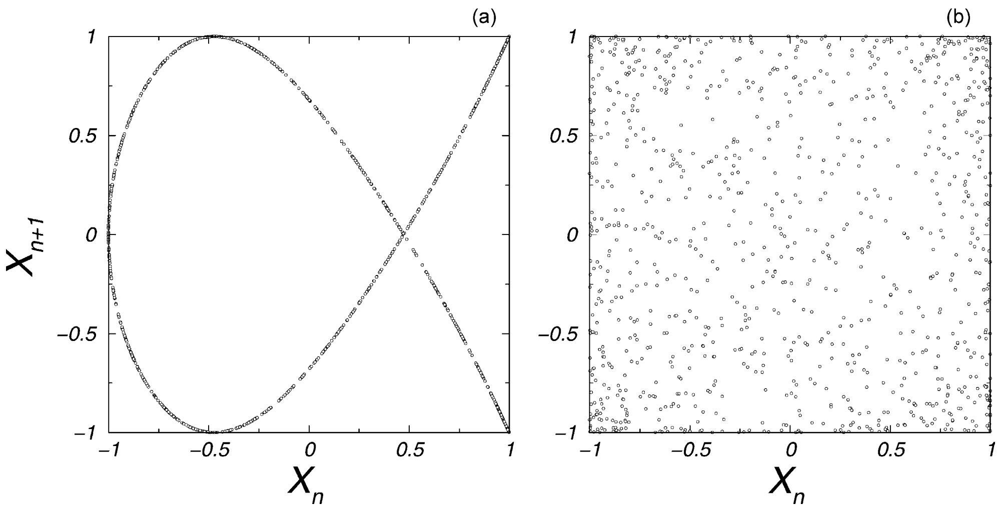

Fig. 3. First-return maps produced by function (12): (a) $z = 3/2$ and (b) $z = \pi/3$.

Interesting first-return maps can be generated by a superposition of several periodic functions $P_1(t), P_2(t), \ldots$ with the argument function $t = \theta z^n$. See, for instance, the maps generated by function

$$X_n = a \sin(c\theta\pi z^n) + b \sin(\mathrm{d}\theta\pi z^n) \tag{14}$$

in Fig. 5.

Note that in all these cases, even if for rational z we can have some structure in the first-return maps, for irrational z the dynamics is structureless.

Another very important stochastic function is the following:

$$X_n = \ln[\tan^2(\theta\pi z^n)]. \tag{15}$$

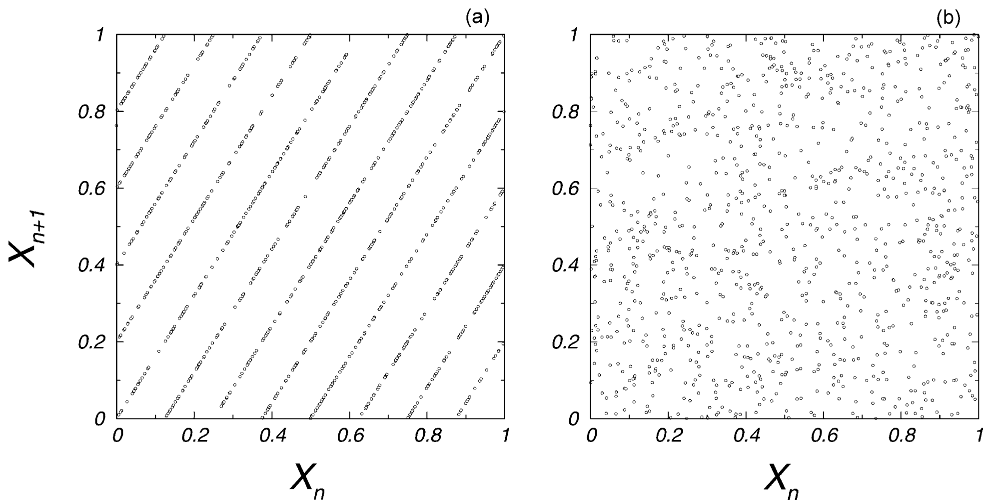

Fig. 4. First-return maps produced by function (13): (a) $z = 8/5$ and (b) $z = e/2$.

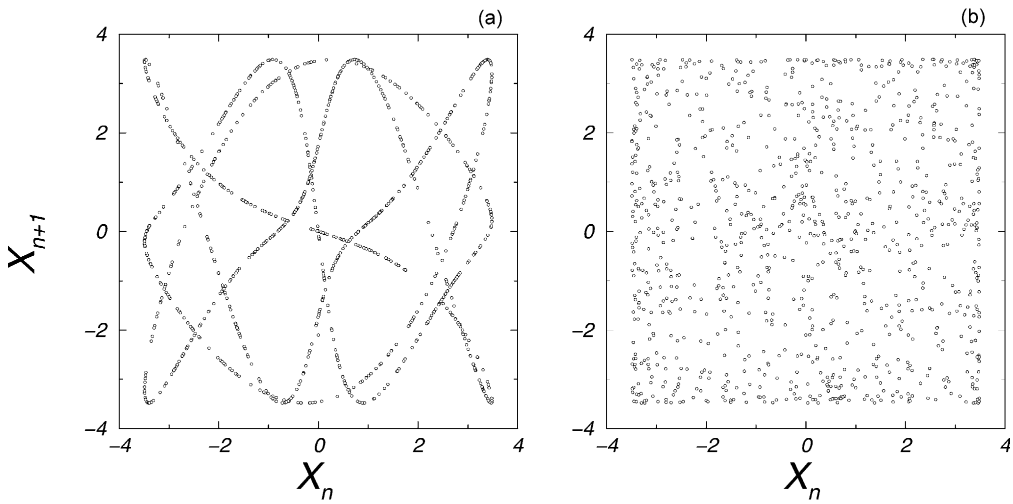

Fig. 5. First-return maps produced by function (14) ($a = 1$, $b = c = 3$, $d = 3/2$): (a) $z = 5/3$ and (b) $z = \pi/3$.

The outcomes produced by this function are distributed following a Gaussian-like law (this can be obtained analytically).

We should say here that in [26], Umeno shows a method of construction of exactly solvable maps using functions of the type:

$$X_n = f[\tan(\theta\pi k^n)], \tag{16}$$

where k is an integer number.

Using these functions, different distributions can be obtained, including the Gaussian law and nonGaussian Levy's laws such as the Cauchy law.

It is interesting that the distribution of the sequences shown in Fig. 6(a) and (b) are the same. However, the actual dynamics is very different. In the case $z = e$, this dynamics is very similar to Gaussian white noise. In the case

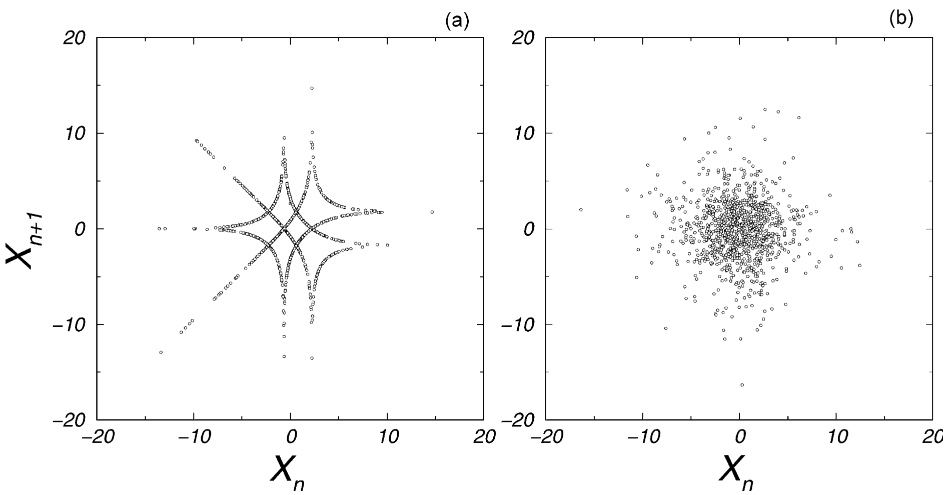

Fig. 6. First-return maps produced by function (15): (a) $z = 5/4$ and (b) $z = e$.

$z = 5/4$, when X_n is close to the point $X_n = 0$, we can say that, for any string of values $X_k, X_{k+1}, X_{k+2}, \ldots, X_{k+m}$, there are always four different possible next values. However, unlike the case $z = e$, for $z = 5/4$, we can know these possible four values for each X_n (although, before the outcome, we will not know which of the four values will be produced). Even more interesting is this: although the distribution is symmetric (the mean value is $X = 0$), the dynamics is not symmetric. For large values of X_n (say $X_n > 10$), we can say that there are two possible next values.

In this case, the uncertainty is less than in a neighborhood of $X_n = 0$. However, for $X_n < -10$, there are always three possible values. And these three values diverge for $X_n \to \infty$.

We will see in Sections 6 and 7 that physical systems can be constructed such that the dynamics of these functions can be realized in practice. In some cases this can be done when we have a many component chaotic system and some of the chaotic dynamical variables are transformed by the nonlinear dynamics of other variables. Using different nonlinearities, different dynamical behaviors can be produced. For instance, we can obtain some of the dynamics shown in Figs. 1–5.

This theory can help to predict the general behavior of some nonlinear systems. Moreover, in special cases some random systems can be more predictable than others.

We should say that a more deep analysis shows that (to produce complex behavior) the function $f(n)$ in (8) does not have to be exponential all the time, and function $h(y)$ does not have to be periodic. In fact, it is sufficient for function $f(n)$ to be a finite nonperiodic oscillating function which possesses repeating intervals with finite exponential behavior. For instance, this can be a chaotic function. On the other hand, function $h(y)$ should be noninvertible. In other words, it should have different maxima and minima in such a way that equation $h(y) = \alpha$ (for some specific interval of α, $\alpha_1 < \alpha < \alpha_2$) possesses several solutions for y.

To conclude this section we will present a random function constructed with a nonperiodic function $P(t)$, where the argument t is defined as $t = \theta z^n$.

Let us define $P(t)$ as

$$P(t) = \sin\{B_1 \sinh[a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t)] + B_2 \cosh[a_3 \cos(\omega_3 t) + a_4 \cos(\omega_4 t)]\}, \tag{17}$$

where $B_1 = 20$, $B_2 = 30$, $a_1 = 10$, $a_2 = 15$, $a_3 = 10$, $a_4 = 15$, $\omega_1 = 1$, $\omega_2 = \pi$, $\omega_3 = \sqrt{2}$, $\omega_4 = e$.

Although this function is composed of quasiperiodic functions, its behavior can be very complicated (see Fig. 7). Fig. 8 shows the dynamics of function $P(\theta z^n)$, where $P(t)$ is given by Eq. (17). We wish here to briefly discuss the role played by the Lyapunov exponent in random chaotic systems.

Suppose we have a random map of type:

$$X_{n+1} = f(X_n, W_n), \tag{18}$$

where W_n is a random variable.

The Lyapunov exponent λ can be defined [27] as the separation rate of two nearby trajectories with the same realization of the random variable W_n. Now suppose that we are also interested in a measure of complexity K in terms of the average number of bits per time unit necessary to specify the sequence generated by the system.

It has been proved [27,28] that K should be calculated as the separation rate of two nearby trajectories using two different realizations of the random variable.

At the same time, there is a relation between K and λ

$$K = \lambda\theta(\lambda) + h, \tag{19}$$

where $\theta(\lambda)$ is the Heaviside step function and h is the Kolmogorov–Sinai entropy of the random variable W_n.

When we calculate the Lyapunov exponent of our function (1), we obtain the following result:

$$\lambda = \ln z \tag{20}$$

for all z.

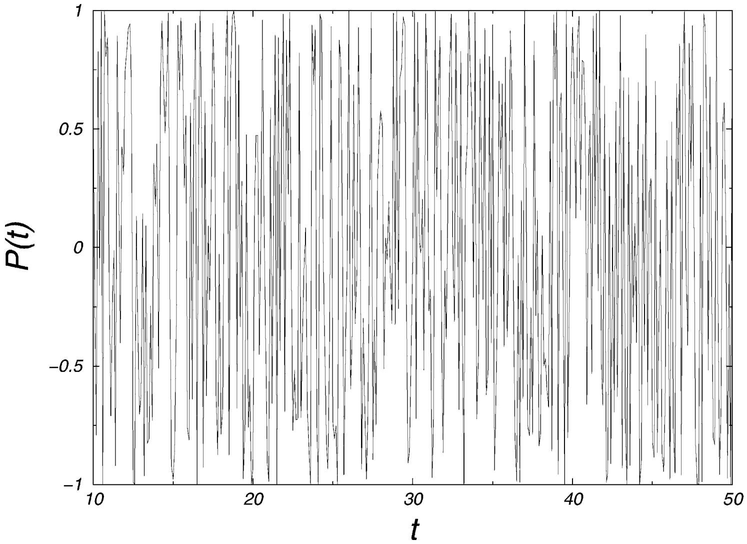

Fig. 7. The function (17) can appear very complex.

If we calculate K as defined before, we obtain

$$K = \ln z \quad (21)$$

for z integer, and

$$K = \ln p \quad (22)$$

for $z = p/q$.

For a usual chaotic map of type $X_{n+1} = f(X_n)$, K coincides with the Kolmogorov–Sinai entropy, i.e. $K = \lambda$.

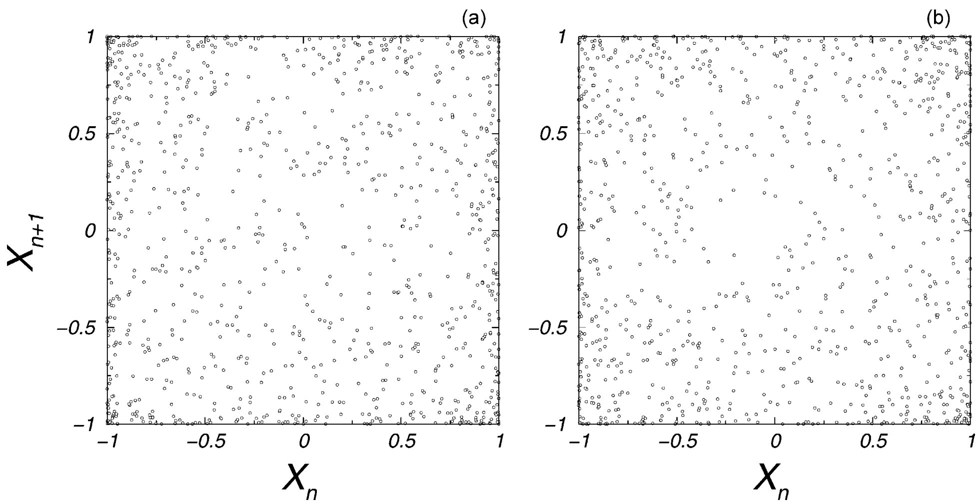

Fig. 8. First-return maps produced by function $P(\theta z^n)$, where $P(t)$ is the function given by Eq. (17): (a) $z = \pi$ and (b) $z = 7/5$. Note that the dynamics is structureless in both cases.

When we have the multivalued maps produced by function (1) with rational $z = p/q$, the difference with the standard chaos is that the information lacking is not given by $K = \lambda$, but is larger: one loses information not only in each iteration due to $\lambda > 0$. One has also to specify the branch of the map (X_n, X_{n+1}).

When the sequences produced by function (1) are numerically analyzed by the algorithm of Wolf et al. [29] and by the method for the calculation of complexity presented in [30], we obtain the same values discussed in this section.

4. Random dynamics generated by an autonomous dynamical system

The following autonomous dynamical system can produce truly random dynamics:

$$X_{n+1} = \begin{cases} aX_n & \text{if } X_n < Q, \\ bY_n & \text{if } X_n > Q, \end{cases} \tag{23}$$

$$Y_{n+1} = cZ_n, \tag{24}$$

$$Z_{n+1} = \sin^2(\pi X_n). \tag{25}$$

Here $a > 1$ can be an irrational number, $b > 1$, $c > 1$. We can note that for $0 < X_n < Q$, the behavior of function Z_n is exactly like that of function (1).

For $X_n > Q$ the dynamics is reinjected to the region $0 < X_n < Q$ with a new initial condition. While X_n is in the interval $0 < X_n < Q$, the dynamics of Z_n is unpredictable as it is function (1). Thus, the process of producing a new initial condition through Eq. (24) is random.

If the only observable is Z_n, then it is impossible to predict the next values of this sequence using only the knowledge of the past values.

An example of the dynamics produced by the dynamical system (23)–(25) is shown in Fig. 9. If we apply the nonlinear forecasting method analysis to a common chaotic system, then the prediction error increases with the number of time-steps into the future. On the other hand, when we apply this method to the time-series produced by system (23)–(25), the prediction error is independent of the time-steps into the future, as in the case of a random sequence. Other very strong methods [29,31] which allow to distinguish between chaos and random noise, produce the same result.

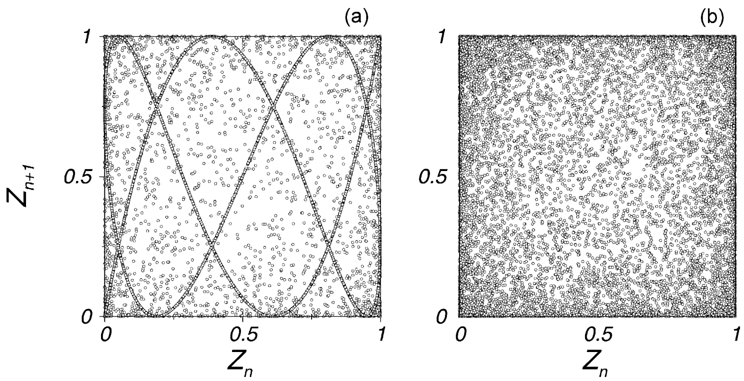

Fig. 9. First-return maps produced by the dynamics of the dynamical system (23)–(25). (a) $a = 7/3$, $b = 171$, $c = 1.5$, $Q = 1000/a$; (b) $a = e$ (irrational), $b = 171$, $c = 1.5$, $Q = 1000/a$.

With this result we are uncovering a new mechanism for generating random dynamics. This is a fundamental result because it is very important to understand different mechanisms by which the natural systems can produce truly random (not only chaotic) dynamics.

5. Other dynamical systems

In the present section we will discuss different dynamical systems, whose dynamics is similar to that of random function (1).

In the previous section we studied a dynamical system where the re-injection of variable X_n to the interval $0 < X_n < Q$ was defined using the random variable Z_n. Here we present a dynamical system as follows:

$$X_{n+1} = \begin{cases} aX_n & \text{if } X_n < Q, \\ bY_n & \text{if } X_n > Q, \end{cases} \tag{26}$$

$$Y_{n+1} = f(Y_n), \tag{27}$$

$$Z_{n+1} = \sin^2(\pi X_n), \tag{28}$$

where the mentioned re-injection is produced through a chaotic system (in this case this is the map $Y_{n+1} = f(Y_n) \equiv \sin^2[c \arcsin\sqrt{Y_n}]$.

Fig. 10 shows two different examples of the dynamics of this system. In order to produce this kind of dynamics we do not need to use the sine function in Eq. (28). We can use a polynomial function:

$$X_{n+1} = \begin{cases} aX_n & \text{if } X_n < Q, \\ bY_n & \text{if } X_n > Q, \end{cases} \tag{29}$$

$$Y_{n+1} = \sin^2[c \arcsin\sqrt{Y_n}], \tag{30}$$

$$Z_{n+1} = h(X_n). \tag{31}$$

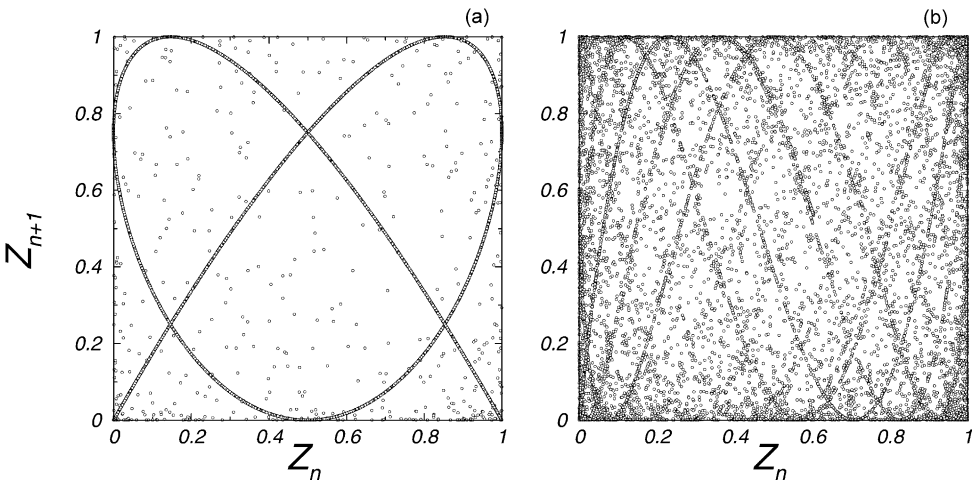

Fig. 10. The map (Z_n, Z_{n+1}), where the dynamics Z_n is produced by the dynamical system (26)–(28): (a) $a = 4/3$, $b = 2$, $c = 3$, $Q = 100$; (b) $a = \pi$, $b = 2$, $c = 3$, $Q = 100$.

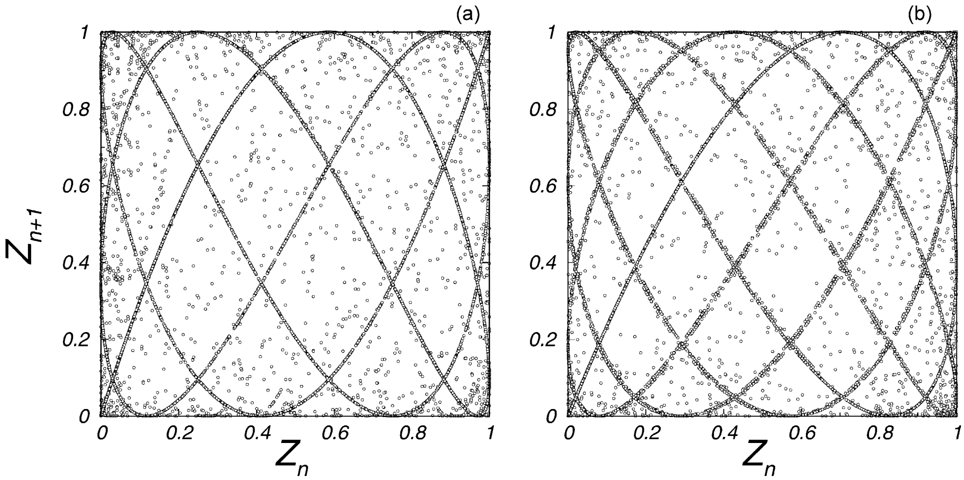

Fig. 11. The map (Z_n, Z_{n+1}), where the dynamics Z_n is produced by the dynamical system (29)–(31), $b = 2.49$, $c = 3$, $Q = 63$, $h(x) = [\sum_{i=1}^{m}((-1)^i(\pi x)^i/i!)]^2$, $m = 90$: (a) $a = 9/5$ and (b) $a = 1.5707963$.

It is important that the function X_n behaves as a truncated exponential "inside" function $h(X_n)$. And function $h(X_n)$ should make several "oscillations" in the interval that coincides with the range of function X_n (i.e. the image of function X_n). For a set of fixed parameters, the number of "oscillations" of function $h(X_n)$ is proportional to the "complexity" of the dynamics. An example of this dynamics can be observed in Fig. 11.

Another way to re-inject the dynamics of function X_n into region $0 < X_n < Q$ is the following:

$$X_{n+1} = \begin{cases} a_1 X_n & \text{if } X_n < Q_1, \\ b_1 W_n & \text{if } X_n > Q_1, \end{cases} \tag{32}$$

$$Y_{n+1} = \sin^2[\pi X_n], \tag{33}$$

$$V_{n+1} = \begin{cases} a_2 V_n & \text{if } V_n < Q_2, \\ b_2 Y_n & \text{if } V_n > Q_2, \end{cases} \tag{34}$$

$$W_{n+1} = \sin^2[\pi V_n]. \tag{35}$$

In this case, we are using the random dynamics produced in a second system to generate the new "initial" conditions for the variable X_n.

See Figs. 12–14 with some examples of the behavior of system (32)–(35). If, due to some physical constraints, the dynamics of a simple chaotic system as Eq. (30) can become a little "repetitive", then the system (32)–(35) is a good way to avoid predictability.

Moreover, we could use a completely independent dynamical system as the one explained in Section 4, to produce the re-injection:

$$X_{n+1} = \begin{cases} a_1 X_n & \text{if } X_n < Q_1, \\ b_1 W_n & \text{if } X_n > Q_1, \end{cases} \tag{36}$$

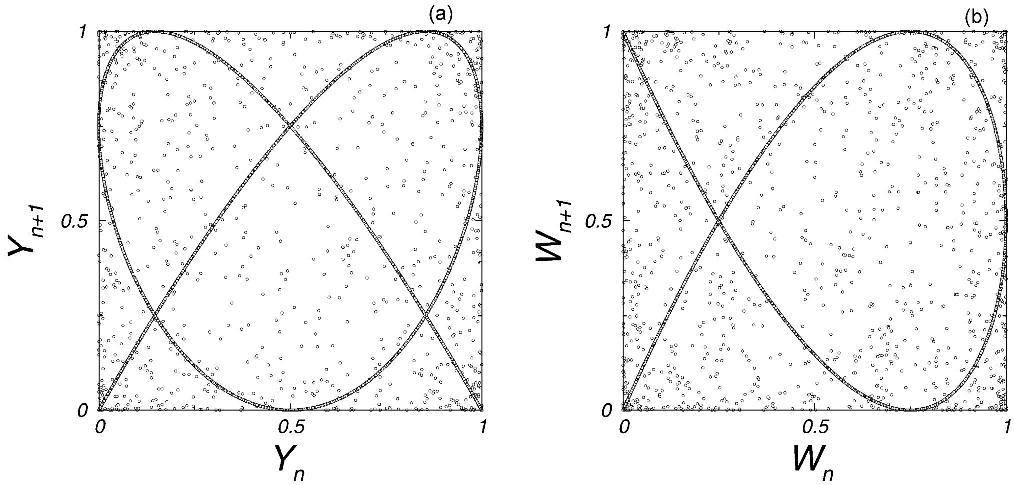

Fig. 12. First-return maps produced by dynamical system (32)–(35), $a_1 = 4/3$, $a_2 = 3/2$, $Q_1 = Q_2 = 950$, $b_1 = b_2 = 249$: (a) map (Y_n, Y_{n+1}) and (b) map (W_n, W_{n+1}).

$$Y_{n+1} = \sin^2[\pi X_n], \tag{37}$$

$$V_{n+1} = \begin{cases} a_2 V_n & \text{if } V_n < Q_2, \\ b_2 W_n & \text{if } V_n > Q_2, \end{cases} \tag{38}$$

$$W_{n+1} = \sin^2[\pi V_n]. \tag{39}$$

Note that Eqs. (38) and (39) are completely independent of Eqs. (36) and (37). Using the same mechanism as that explained in Section 4, the dynamics of variable W_n is very complex. This dynamics is used in Eq. (36) to produce the new "initial" conditions for X_n.

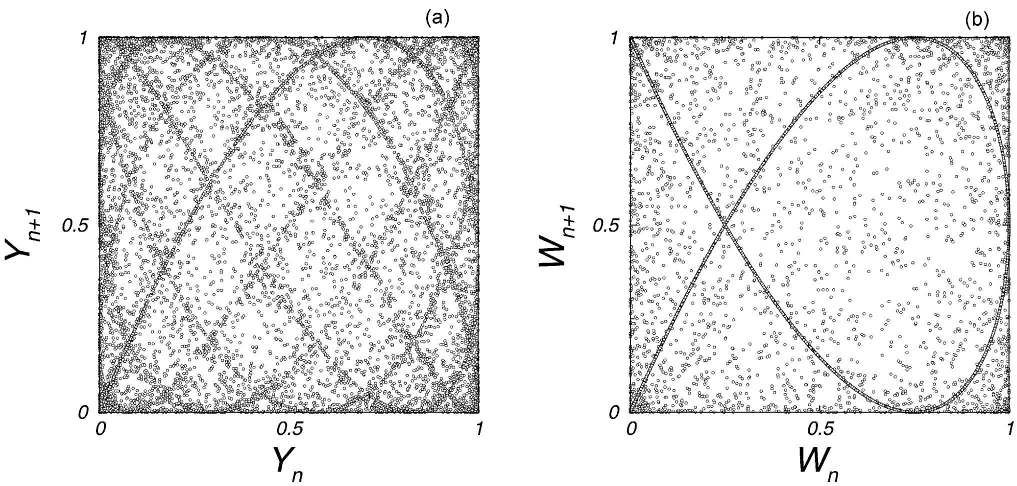

Fig. 13. First-return maps produced by dynamical system (32)–(35), $a_1 = 1.5707963$, $a_2 = 3/2$, $b_1 = 3$, $b_2 = 249$, $Q_1 = 95$, $Q_2 = 950$: (a) map (Y_n, Y_{n+1}) and (b) map (W_n, W_{n+1}).

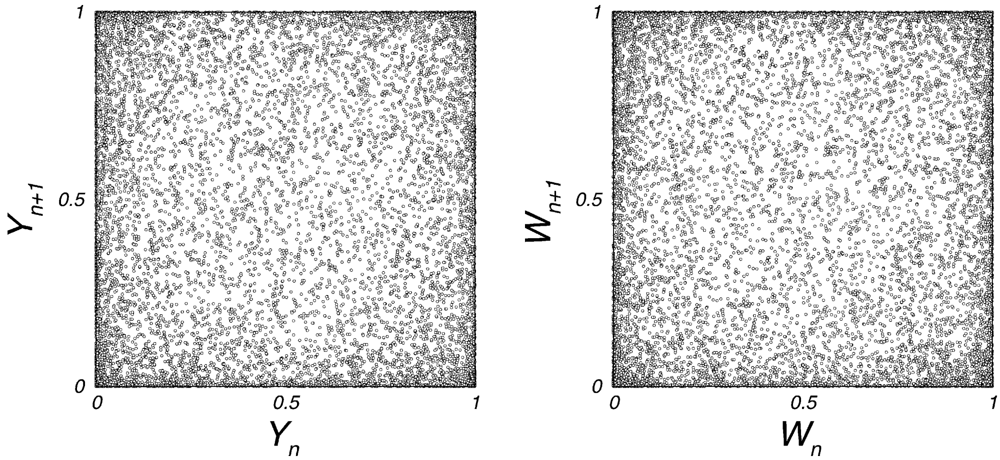

Fig. 14. First-return maps produced by dynamical system (32)–(35), $a_1 = 2\pi$, $a_2 = 2e$, $b_1 = 29$, $b_2 = 24$, $Q_1 = Q_2 = 99$: (a) map (Y_n, Y_{n+1}) and (b) map (W_n, W_{n+1}).

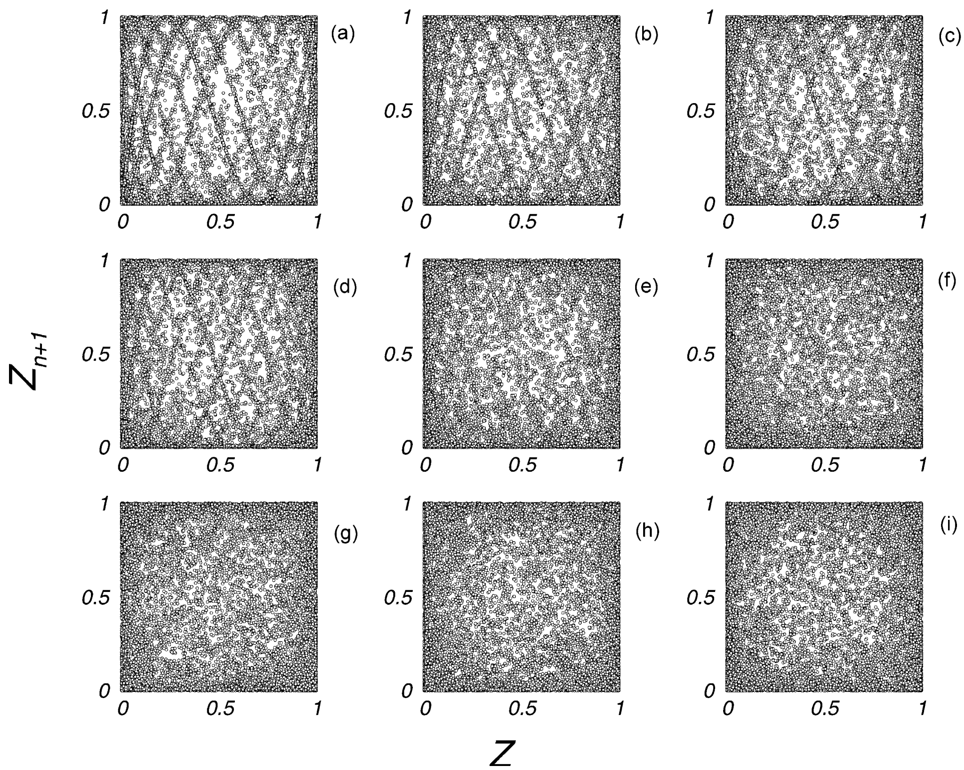

Fig. 15. Maps (Z_n, Z_{n+1}) produced by the dynamical system (40)–(42) (see discussion in the main text): (a) $\nu = 0.5$, (b) $\nu = 1$, (c) $\nu = 1.5$, (d) $\nu = 2$, (e) $\nu = 5$, (f) $\nu = 20$, (g) $\nu = 50$, (h) $\nu = 500$ and (i) $\nu = 5000$. Note that the complexity increases with ν.

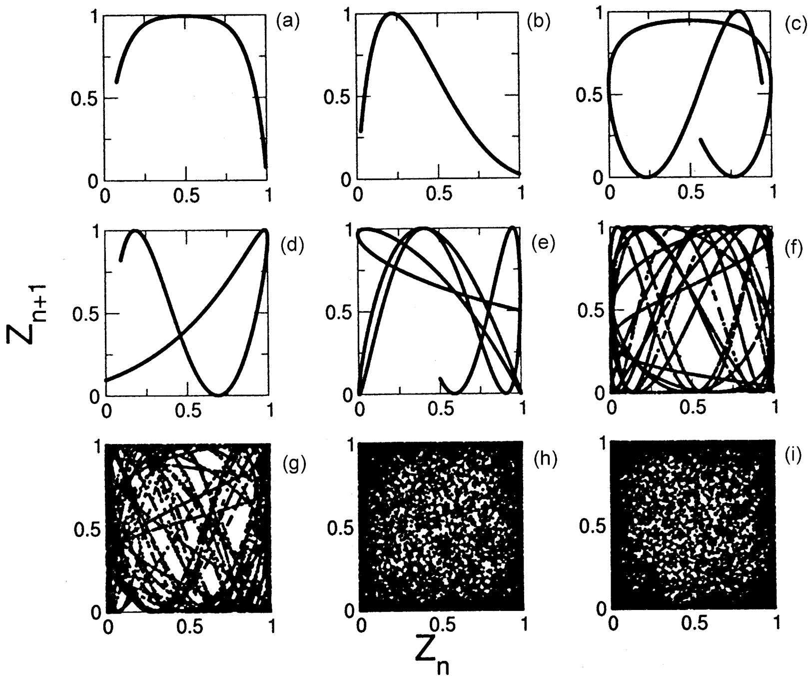

Fig. 16. Maps (Z_n, Z_{n+1}) produced by the dynamical system (40)–(42) (see discussion in the main text): (a) $\nu = 0.5$, (b) $\nu = 1$, (c) $\nu = 1.5$, (d) $\nu = 2$, (e) $\nu = 5$, (f) $\nu = 20$, (g) $\nu = 50$, (h) $\nu = 500$ and (i) $\nu = 5000$. Note that in this case the dynamics produced by Eqs. (40) and (41) is very simple (predictable) chaotic signal, limited to the interval $0 < X_n < 1$. Even in this case, increasing parameter ν, the complexity can be increased to unimaginable values.

Now we would like to discuss the role of noninvertibility in the creation of complex dynamics. Let us study dynamical systems of the type:

$$X_{n+1} = f(X_n, Y_n), \tag{40}$$

$$Y_{n+1} = g(X_n, Y_n), \tag{41}$$

$$Z_{n+1} = \sin^2(\pi \nu X_n). \tag{42}$$

Suppose Eqs. (40) and (41) produce chaotic dynamics where there is intermittent finite exponential behavior.

Note that in the argument of the sine function in Eq. (42) there is a new parameter ν. Using this parameter we can change the number of oscillations of the sine function in a given interval of possible values of variable X_n.

If Eqs. (40) and (41) conform a dynamical system of type (26) and (27), where there exists a parameter Q, which defines approximately the maximum value of variable X_n, then we can increase this value at our will, and it is not difficult to produce complex dynamics.

In any case, parameter ν can help to produce even more complex dynamics (see Fig. 15). However, suppose the dynamical system (40) and (41) can produce a chaotic (but predictable) dynamics limited to the interval $0 < X_n < 1$ like the logistic map. Can we obtain very complex dynamics in Eq. (42)?

In Fig. 16 we can observe the consequences of increasing parameter ν. Thus, in a chaotic system as the following:

$$X_{n+1} = f(X_n, Y_n), \tag{43}$$

$$Y_{n+1} = g(X_n, Y_n), \tag{44}$$

$$Z_{n+1} = h(X_n), \tag{45}$$

the number of "oscillations" of function $h(X_n)$ in the interval of possible values of the chaotic variable X_n is crucial for producing complexity.

6. Nonlinear circuits

When the input is a normal chaotic signal and the system is an electronic circuit with the I–V characteristics shown in Figs. 17 and 18, then the output will be a very complex signal.

In [32] a theory of nonlinear circuits is presented. There we can find different methods to construct circuits with these I–V characteristic curves.

The scheme of this composed system is shown in Fig. 19. A set of equations describing this dynamical system is the following:

$$X_{n+1} = F_1(X_n, Y_n), \tag{46}$$

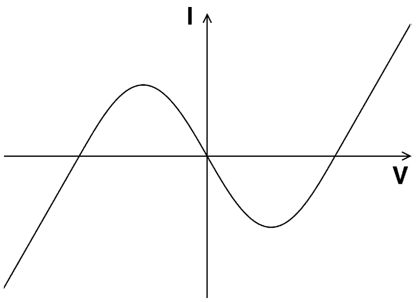

Fig. 17. Noninvertible I–V characteristic. Two extrema.

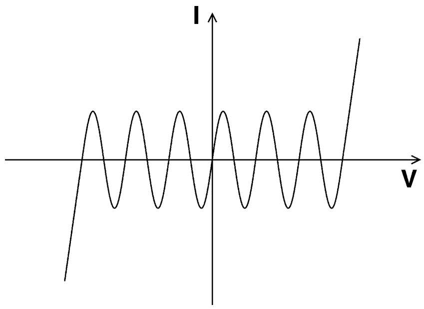

Fig. 18. Noninvertible I–V characteristic. Many extrema.

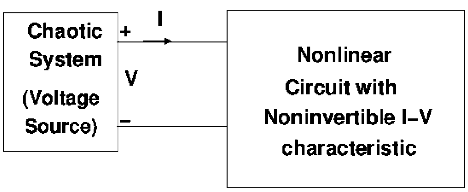

Fig. 19. Scheme of a nonlinear system where a chaotic voltage source is used as the input signal for a nonlinear circuit with a noninvertible I–V characteristic.

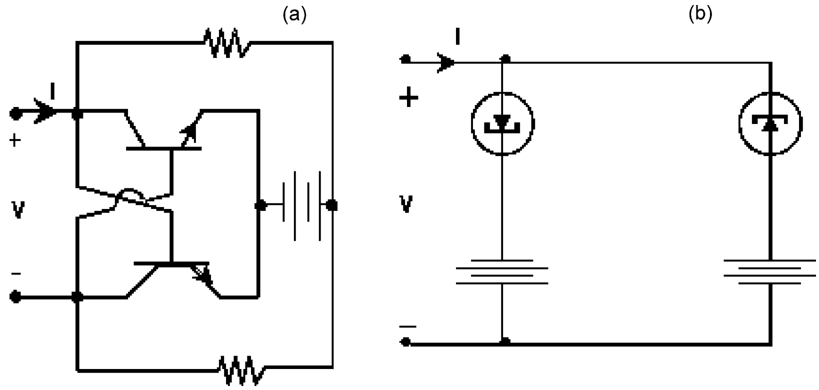

Fig. 20. Nonlinear circuits with noninvertible I–V characteristics. (a) The resistors possess $R = 2.2\,\text{k}\Omega$, the source voltage in the battery is 10 V and the twin transistors are $2N2222$ with $\beta = 140$. The I–V characteristic of this circuit is shown in Fig. 4. (b) Another circuit with a similar I–V characteristic.

$$Y_{n+1} = F_2(X_n, Y_n), \tag{47}$$

$$Z_{n+1} = g(X_n), \tag{48}$$

where Eqs. (46) and (47) describe a normal chaotic dynamics where the variable X_n presents intermittent intervals with a truncated exponential behavior and $g(X_n)$ is a function with several maxima and minima as that shown in Fig. 18.

Fig. 20(a) and (b) show nonlinear circuits that can be used as the nonlinear system shown on the right of the scheme of Fig. 19. The system on the left of the scheme can be a chaotic circuit, e.g. the Chua's circuit [33]. We have constructed a circuit similar to the one shown in Fig. 20(a). We produced chaotic time-series using a common nonlinear map and then we transformed them into analog signals using a converter. This analog signals were introduced as the voltage-input to the circuit shown in Fig. 20(a). Similar results are obtained when we take the input signal from a chaotic electronic circuit.

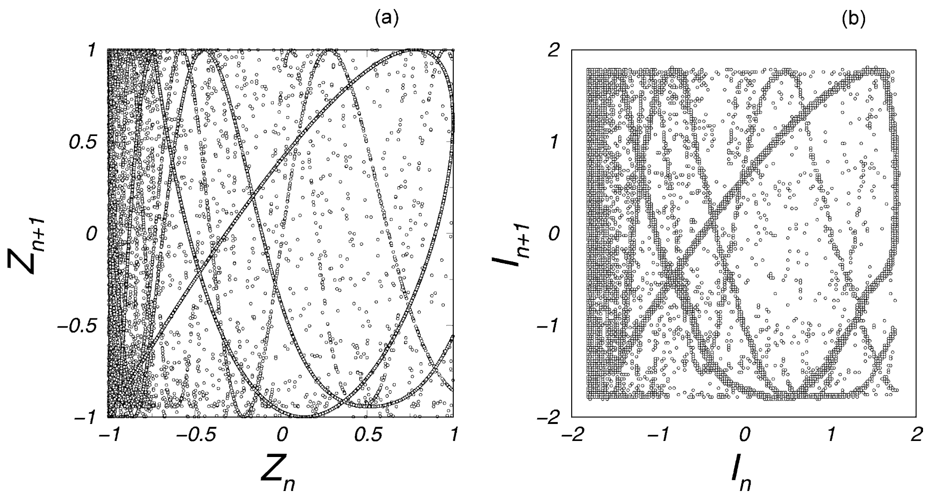

Fig. 21. Modeling vs. experiment. (a) Numerical simulation of the dynamical system (49)–(51); (b) first-return map produced with the real data (current measurements) from experiments using the scheme of Fig. 6, where the circuit of the right is the one of Fig. 7(a).

The set of equations that describes one of our experimental situations is the following:

$$X_{n+1} = aX_n[1 - \Theta(X_n - q)] + bY_n\Theta(X_n - q), \quad (49)$$

$$Y_{n+1} = \sin^2[d \arcsin\sqrt{Y_n}], \quad (50)$$

$$Z_{n+1} = 4W_n^3 - 3W_n, \quad (51)$$

where $W_n = (2X_n/s) - 1$, $q = s/a$, $s = 10$, $b = 7$, $a = \pi/2$, $d = 3$, $\Theta(x)$ is the Heaviside function.

The first-return maps of the sequence Z_n produced by the theoretical model (49)–(51) and the experimental time-series produced by the nonlinear system of Figs. 19 and 20(a) are shown in Fig. 21(a) and (b).

When the nonlinear circuit has an I–V characteristic with many more maxima and minima, e.g., Fig. 18 (and this can be done in practice, see [32]), we can produce a much more complex dynamics.

7. Josephson junction

Using our theoretical results we can make a very important prediction here. A nonlinear physical system constructed with chaotic circuits and a Josephson junction [34] can be an ideal experimental setup for the random dynamics that we are presenting here.

It is well-known that the current in a Josephson junction may be written as

$$I = I_c \sin\phi, \quad (52)$$

where

$$\frac{d\phi}{dt} = kV. \quad (53)$$

Here ϕ is the phase difference of the superconducting order parameter between each side of the barrier and V is the voltage across the junction. Note that Nature has provided us with a phenomenon where the sine-function is intrinsic. Although we have already explained that other noninvertible functions can produce similar results, it is remarkable that we can use this very important physical system to investigate the real consequences of our results with function (1). In a superconducting Josephson junction k is defined through the fundamental constants $k = 2e/\hbar$. However, in the last decades there have been a wealth of experimental work dedicated to the creation of electronic analogs that can simulate the Josephson junction [35–38]. In that case k can be a parameter with different numerical values.

We have performed real experiments with a nonlinear chaotic circuit coupled to an analog Josephson junction. In our experiments we have used the Josephson junction analog constructed by Magerlein [38]. This is a very accurate device that has been found very useful in many experiments for studying junction behavior in different circuits. The junction voltage is integrated using appropriate resetting circuitry to calculate the phase ϕ, and a current proportional to $\sin\phi$ can be generated. The circuit diagram can be found in [38].

The parameter k is related to certain integrator time constant RC in the circuit. So we can change its value. This is important for our experiments. We need large values of k in order to increase the effective domain of the sine function. In other words, we need the argument of the sine function to take large values in a truncated exponential fashion. This allows us to have a very complex output signal. In our case the value of k is 10,000.

The voltage $V(t)$ across the junction is not taken constant. This voltage will be produced by a chaotic system. In our case we selected the Chua's circuit [33]. For this, we have implemented the Chua's circuit following the recipe of [39]. The scheme of the Chua's circuit constructed by us can be found in Fig. 1 of [39]. The following components were used: $C_1 = 10$ nF, $C_2 = 100$ nF, $L = 19$ mH and R is a 2.0 kΩ trimpot.

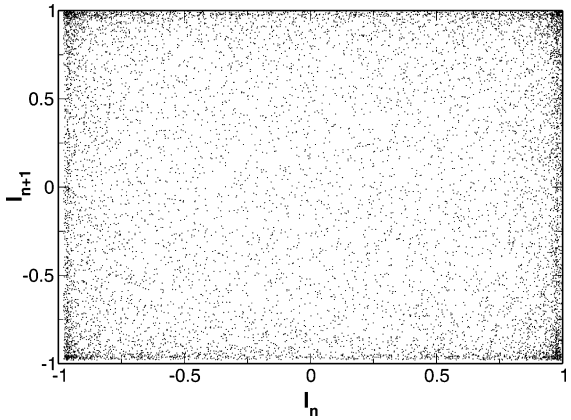

Fig. 22. First-return map of the time-series generated with real data from an experiment with an analog Josephson junction coupled to the Chua's circuit.

Chua's diode was built using a two-operational-amplifier configuration suggested in [39]. In our experiment, the voltage in C_1 was used as the driving signal for the Josephson junction. We were interested in the famous double scroll attractor attained with $R \approx 1880\,\Omega$.

The results of the experiments are shown in Fig. 22 which is the first-return map of the time-series data produced by direct measurements of the junction current. The time intervals between measurements was 10 ms. This system can produce unpredictable dynamics.

We would like to add here that there are other physical systems where functions of type $X_n = P(\theta z^n)$ can be implemented. See for instance [40] where an all-optical method of constructing exactly solvable chaos is presented based on $\sin^2(x)$ functions made by Nature (this method uses Mach–Zehnder interferometers).

8. Computation and complexity

Recently, there have been important developments in dynamics-based computation [41–44]. In [41] it is demonstrated that the ability of lattices of coupled chaotic maps to perform simple computations. These chaotic systems can emulate logic gates, encode numbers, and perform specific arithmetic operations on those numbers such as addition and multiplication.

Many of the applications of chaos (including the chaos-based computation) are based on the flexibility that chaos provides in the performance of natural systems. As it is discussed in [41], these chaotic systems possess a rich repertory of behaviors that can be utilized for improved performance.

Considering the fact that the dynamical systems studied in this paper possess a dynamics richer than that of the normal chaotic systems, we believe that the new systems can be used to perform more complicated operations.

In fact, our systems possess all the advantages of random systems with the richest repertory of behaviors. At the same time, the dynamics can be controlled to perform the desired operations. It is possible to design an encoding scheme that exploits the randomness of the dynamics generated by systems as the one defined by Eqs. (23)–(28), (32)–(39). A dynamical system able to produce randomness can yield all the possible sequences, which is an advantage (in the process of encoding numbers) over other chaotic systems. Another important area of research in the field of dynamics-based computation is the so-called "computation beyond the Turing limit" [42–44].

Behind this effort are new chaotic systems called "generalized shifts". The first report about a generalized shift can be found in [42]. In order to introduce the generalized shifts, let us define two-sided infinite sequences:

$$a = (a_i) = \cdots a_{-2}a_{-1} \cdot a_0a_1a_2 \cdots . \tag{54}$$

A normal horseshoe map is equivalent to a shift map $\sigma : a_i \to a_{i+1}$. In this map there is a chaos because errors grow exponentially as differences in the symbol sequence are shifted towards the origin.

In general, to predict the system s steps in the future, we need s digits of the initial conditions. If we need to calculate the ith digit s steps in the future, then we need to look at the $(i + s)$th digit. The generalized shift map is defined as follows $\phi : a \to \sigma^{F(a)}(a \oplus G(a))$. Here F is a map from a to the integers, and G is a map from a to finite sequences. The procedure is the following: replace a finite number of cells in a with the sequence $G(a)$. Then shift the sequence left or right by the amount $F(a)$. An additional requirement is that F and G depend on a finite number of cells in a.

This system is equivalent to a Turing machine, and so is capable of universal computation. The generalized shifts are more complex than the chaotic systems that are equivalent to the common shift. Even if the initial conditions are known, virtually any question about their long-term dynamics is undecidable. In the example discussed in [42], the dynamics appears to be ergodic. However, the divergence of close initial conditions is slower than exponential.

A generalization of these systems called "analog shift map" is used in [43] to discuss the possibility that a machine can have computational power beyond the Turing limit. Computers designed and built on the basis of super-Turing theories should be capable of modeling phenomena that existing computers are not powerful enough to model in a satisfactory way.

In computer science, machines are classified according to the classes of tasks they can execute or the function they can perform. The finiteness of the input and output is a crucial requirement in a computer science, assuring that the ability to compute depends purely on the inherent computational power of the model rather than on a large amount of external information.

Siegelman presents a chaotic dynamical system that computationally is as strong as the analog computational model. This chaotic dynamical system is called the "analog shift map". Chaos is usually modeled by the shift map. Let E be a finite alphabet. A dotted sequence over E (denoted by $\dot{E}$) is a sequence of letters where exactly one is the dot sign $(\cdot)$ and the rest are all in E.

Let $k \in N$. A new shift map can be defined as

$$S^K : \dot{E} \to \dot{E} : (a_i) \to (a_{i+k}). \tag{55}$$

This map shifts the dot k places, where negative values cause a shift to the left and positive ones a shift to the right.

Siegelman defines the "analog shift" as follows: a dotted substring is replaced with another dotted substring (of equal length) according to a function G; then, this new sequence is shifted an integer number of places left or right according to a function F.

Formally, the analog shift is the map

$$\phi : a \to s^{F(a)}(a \oplus G(a)), \tag{56}$$

where the function $G : \dot{E} \to \dot{E}$ describes the modification of the sequence, and the function $F : \dot{E} \to Z$ indicates the amount of shifting of the dot.

Both F and G have a finite domain of dependence, that is, F and G depend only on the finite dotted substring of the sequence on which they act. The domain of effect of G may be finite, infinite or bi-infinite.

The operation $\oplus$ is defined by

$$(a \oplus g)_i = \begin{cases} g_i & \text{if } g_i \in E, \\ a_i & \text{if } g_i = \varepsilon, \end{cases} \tag{57}$$

where ε is the empty element not contained in E.

When the domain of effect of G is finite we obtain the "generalized shift" (introduced by Moore [42]), which was proven to be computationally equivalent to a Turing machine. When G is not applied, we obtain the common shift maps.

The computation associated with the analog shift systems is the evolution of the initial dotted sequence until a fixed point is reached, from which the system does not evolve anymore. When the computation ends, the input–output map is defined as the transformation from the initial dotted sequence to the final subsequence to the right of the dot. Even under the computational constraints of finite input–output, the analog shift is said to compute richer maps than the Turing machines [43].

Physical systems exhibit various levels of complexity: their long-term dynamics may converge to periodic attractors or exhibit complex chaotic behavior. Natural processes can be interpreted as special purpose analog computers. The input for the machine is the initial condition for the dynamical system. The series of consecutive configurations during the computation process define the trajectory in the space of machine configurations.

The correspondence between computers and physical systems allows us to view the evolution of physical systems as a process of computation. Physical systems are in fact analog computers. There seems to be an algorithmic advantage in using analog computers for specific problems.

Analog computation can be utilized to test possible limitations of the physical Church-Turing thesis [43,45] that states that the computational capabilities of any physical device cannot exceed that of a Turing machine.

If a device that computes problems that cannot be computed by the Turing machine (and therefore by digital computers) is found, it will challenge the physical Church-Turing thesis. Some theoretical analog models of computation seem to have the capability of computing beyond the Turing limit [43], but no realizable super-Turing system has been found.

The trajectory of a dynamical system can be thought as describing a computation, or equivalently solving a computational problem. The initial state is the input of a computational problem, the evolution along the trajectory is the computation process, and the attractor describes the solution.

A computing device maps an input into an output, the internal evolution is regarded as the computation. For dynamical systems, the initial condition corresponds to the input and the system evolves until approaching an attractor which represents the output.

In [45] the focus of the analysis are dynamical systems with exponential convergence to the attractors. In fact, the complexity of the computational process is defined in terms of the total computation time.

Analog computation is very important in a wide range of engineering applications such as control and robotics. Can we find a realizable super-Turing dynamical system? As has been stated in the works [42–45], the complexity generated by a dynamical system is crucial for the computation process.

We should say here that although long-term average quantities like escape rates, Lyapunov exponents, or the measure of a basin of attraction are impossible to compute for the generalized shifts, these maps are not as random as the random-chaos systems discussed in the present paper. In fact, in the generalized shifts we talk about unpredictability of the long-term behavior. The sequence Z_n produced by random-chaos systems are unpredictable in a stronger way. When we have a string of previous values $Z_0, Z_1, Z_2, \ldots, Z_n$, even the next value is unpredictable.

We believe that the random-chaos systems can compute richer dynamics that the Turing machines and the generalized shifts.

Besides, the physical realization of the generalized shifts have presented several problems [42,43]. On the other hand, we have seen in the present paper that very practical electronic systems can produce random-chaos dynamics.

What is the complexity produced by the known chaotic systems starting with finite inputs? One of the most important paradigms of chaos is the Bernoulli shift:

$$X_{n+1} = 2X_n \pmod 1. \tag{58}$$

However, this map cannot produce complexity if the initial condition is a rational number. Suppose $X_0 = 0.1$, then we have the string $X_0 = 0.1$, $X_1 = 0.2$, $X_2 = 0.4$, $X_3 = 0.8$, $X_4 = 0.6$, $X_5 = 0.2$ (after that the string is periodic). This happens for any rational number.

Moreover, we can see this phenomenon in a clearer way if we write the numbers X_n in a binary base. For instance

$$X_0 = 0 \cdot a_0 a_1 a_2 a_3 \cdots , \tag{59}$$

where $a_i = \{0, 1\}$.

The next values are obtained shifting the dot: $X_1 = 0 \cdot a_1 a_2 a_3 \cdots$, $X_2 = 0 \cdot a_2 a_3 a_4 \cdots$, $X_3 = 0 \cdot a_3 a_4 a_5 \cdots$, $X_4 = 0 \cdot a_4 a_5 a_6 \cdots$.

We will have chaos only if the initial condition is an irrational number. For example, take $X_0 = 0.11010001$. The dynamics that follows is $X_1 = 0.1010001$, $X_2 = 0.010001$, $X_3 = 0.10001$, $X_4 = 0.0001$, $X_5 = 0.001$, $X_6 = 0.01$, $X_7 = 0.1$, $X_8 = 0$, $X_9 = 0$ (after this, all the next values are zeroes).

This is true for all the shifts, including the generalized shifts and the analog shifts. In order to have chaos and complexity, we need infinite sequences, that is, infinite inputs.

Is there a map of type

$$X_{n+1} = aX_n \pmod 1 \tag{60}$$

that can produce complexity starting with a rational initial condition? The answer is yes. Parameter a must be a noninteger number.

Take for instance $a = 1.57$ and $X_0 = 0.1$. We will have the following string: $X_1 = 0.157$, $X_2 = 0.24649$, $X_3 = 0.386989$, $X_4 = 0.607573\ldots$, $X_5 = 0.95389\ldots$, $X_6 = 0.497607\ldots$, $X_7 = 0.781243\ldots$, $X_8 = 0.226552\ldots$. For $n \to \infty$, X_n will become an irrational number.

Is this a random sequence? No, this is not a random sequence. This is just a normal chaotic sequence. Each value X_{n+1} is completely determined by the previous value. The only advantage with respect to the Bernoulli shift is that we can produce chaos starting with a rational initial condition.

Can we create a dynamical system, able to produce random sequences starting with rational initial conditions? The dynamical systems discussed in this paper in Sections 4 and 5 are able to do that.

An example can be the following:

$$X_{n+1} = \{(a + bZ_n)X_n + cY_n + 0.1\} \pmod 1, \tag{61}$$

$$Y_{n+1} = (dZ_n + 0.1) \pmod 1, \tag{62}$$

$$Z_{n+1} = fX_n \pmod 1. \tag{63}$$

We can use rational initial conditions as the following:

$$X_0 = Y_0 = Z_0 = 0.1, \tag{64}$$

and the parameters $a = 2.57$, $b = 1.97$, $c = 3.02$, $d = 2.09$, $f = 1787$.

The dynamics of variable Z_n is completely random for the same reasons stated in the proof discussed in Section 2 [Eqs. (5)–(7)] and the discussion of dynamical system (26)–(28).

In the specific case of the initial condition (64), we have: $Z_0 = 0.1$, $Z_1 = 0.7$, $Z_2 = 0.8369$, $Z_3 = 0.785578$, $Z_4 = 0.993614\ldots$, $Z_5 = 0.309763\ldots$, $Z_6 = 0.633807\ldots$, $Z_7 = 0.265490\ldots$, $Z_8 = 0.814266\ldots$, $Z_9 = 0.085232\ldots$, $Z_{10} = 0.465068\ldots$, $Z_{11} = 0.878017\ldots$, $Z_{12} = 0.682764\ldots$, $Z_{13} = 0.251184\ldots$, $Z_{14} = 0.880502\ldots$, $Z_{15} = 0.429605\ldots$, $Z_{16} = 0.319685\ldots$, $Z_{17} = 0.042913\ldots$, $Z_{18} = 0.049904\ldots$, $Z_{19} = 0.978487\ldots$, $Z_{20} = 0.970210\ldots$, $Z_{21} = 0.907481\ldots$, $Z_{22} = 0.354334\ldots$.

Now suppose that we have the finiteness constraint also for the output. That is, the output should be given as a finite sequence of numbers. In the language of our dynamical systems, this means that the initial conditions should be given by numbers with a finite number of decimal digits and the numbers produced by the mathematical operations for X_n, Y_n, Z_n, should be defined also with a finite number of decimal digits.

Let us take a very strict example: we are allowed to operate only with one meaningful number after the dot, that is, numbers of type $0.a$.

Can we produce complexity with this limitation? If we return to the map defined in Eq. (60): $X_{n+1} = aX_n \pmod 1$, we will see that (with this constraint) this map cannot produce complexity. Let us define, for instance, $a = 2.7$, $X_0 = 0.1$, then $X_1 = 0.2$, $X_2 = 0.5$, $X_3 = 0.3$, $X_4 = 0.8$, $X_5 = 0.1$ (after that the sequence is periodic).

Now we will present a different map:

$$X_{n+1} = [(a + bZ_n)X_n + cY_n + 0.1] \pmod 1, \tag{65}$$

$$Y_{n+2} = [(Y_{n+1} + Y_n) \pmod 1) + C_n + dZ_n + 0.1] \pmod 1, \tag{66}$$

$$Z_{n+1} = (fX_n + eY_n + 0.1) \pmod 1, \tag{67}$$

$$C_{n+1} = F(C_0, C_1, C_2, \ldots, C_n). \tag{68}$$

Eq. (68) for C_n does not need to be a chaotic map. The sequence C_n can be just a nonperiodic time-series. For example, this can be a quasiperiodic time-series.

The map that produces C_n can be, for instance, a nonperiodic sequence as the one obtained using the digits of the Champernowne's number [46]: $C_0 = 0.1$, $C_1 = 0.2$, $C_2 = 0.3$, $C_3 = 0.4$, $C_4 = 0.5$, $C_5 = 0.6$, $C_6 = 0.7$, $C_7 = 0.8$, $C_8 = 0.9$, $C_9 = 0.1$, $C_{10} = 0.0$, $C_{11} = 0.1$, $C_{12} = 0.1$, $C_{13} = 0.1$, $C_{14} = 0.2$, $C_{15} = 0.1$, $C_{16} = 0.3$, $C_{17} = 0.1$, $C_{18} = 0.4$, $C_{19} = 0.1$, $C_{20} = 0.5, \ldots$.

Although the dynamics of C_n can be quasiperiodic, the dynamics of Z_n is completely random, even if both the input and the output are finite. Here we present an example of a time-series obtained using this procedure. Consider the following initial conditions:

$$X_0 = Y_0 = Z_0 = 0.1,\ Y_1 = 0.2 \tag{69}$$

and the following parameters

$$a = 2.7,\ b = 1.9,\ c = 3.2,\ d = 2.9,\ e = 1.1,\ f = 1984. \tag{70}$$

The sequence Z_n has the following first 32 values: $Z_0 = 0.1$, $Z_1 = 0.6$, $Z_2 = 0.1$, $Z_3 = 0.4$, $Z_4 = 0.6$, $Z_5 = 0.5$, $Z_6 = 0.0$, $Z_7 = 0.1$, $Z_8 = 0.8$, $Z_9 = 0.5$, $Z_{10} = 0.6$, $Z_{11} = 0.9$, $Z_{12} = 0.6$, $Z_{13} = 0.4$, $Z_{14} = 0.3$, $Z_{15} = 0.3$, $Z_{16} = 0.4$, $Z_{17} = 0.1$, $Z_{18} = 0.0$, $Z_{19} = 0.1$, $Z_{20} = 0.8$, $Z_{21} = 0.7$, $Z_{22} = 0.4$, $Z_{23} = 0.7$, $Z_{24} = 0.2$, $Z_{25} = 0.0$, $Z_{26} = 0.3$, $Z_{27} = 0.1$, $Z_{28} = 0.2$, $Z_{29} = 0.2$, $Z_{30} = 0.4$, $Z_{31} = 0.3, \ldots$.

All the strings of m values are possible. And after any string of m values (m is any integer) the next value can be any of the following values: 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9. Besides, the distribution is uniform for $n \to \infty$.

In the context of computation [41–45], the most important result is that these dynamical systems can compute the richest possible maps and functions. On the other hand, we believe that using Josephson junctions the analog computers based on these dynamical systems can be realized in a satisfactory way.

9. Discussion and conclusions

In conclusion, we have shown that functions of type $X_n = P(\theta z^n)$, where $P(t)$ is a periodic function and z is a noninteger number, can produce completely random numbers. Certain class of autonomous dynamical systems can generate a similar dynamics. This dynamics presents fundamental differences with the known chaotic systems. We have presented real nonlinear systems that can produce this kind of random time-series. We have reported the results of real experiments with nonlinear circuits containing direct evidence supporting this phenomenon.

Now we will analyze very general ideas. Just to facilitate our discussion (because it is always important to have a name), we will call the phenomenon studied in this paper "random chaos".

Random chaos imposes fundamental limits on prediction, but it also suggests that there could exist causal relationships where none were previously suspected. Random chaos demonstrates that a system can have the most complicated behavior that emerges as a consequence of simple, nonlinear interaction of only a few effective degrees of freedom.

On one hand, random chaos implies that if there is a phenomenon in the World (whose mechanism from first principles is not known) described by a dynamical system of type [8–10] or [12–14], and the only observable is a physical variable as Z_n, then the law of this phenomenon cannot be learnt from the experimental data, or the observations. And, situations in which the fundamental law should be inferred from the observations alone have not been uncommon in physics.

On the other hand, the fact that this random dynamics is produced by a relatively simple, well-defined autonomous dynamical system implies that many random phenomena could be more predictable than have been thought.

Suppose there is a system thought to be completely random. From the observation of some single variable, scientists cannot obtain the generation law. However, suppose that in some cases, studying the deep connections of the phenomenon, we can deduce a dynamical systems of type [8–10] or [12–14]. In these cases, some prediction is possible. Earthquakes and stock price dynamics have eluded prediction theorists until now. Who knows if these phenomena could be examples of random chaos. In the present paper we have also discussed the possible applications of random chaos in dynamics-based computation.

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